GENERALIZATIONS OF MCSHANE'S IDENTITY TO HYPERBOLIC CONE-SURFACES

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Abstract. We generalize McShane's identity for the length series of simple closed geodesics on a cusped hyperbolic surface [17] to hyperbolic cone-surfaces (with all cone angles $\leq \pi$), possibly with cusps and/or geodesic boundary. In particular, by applying the generalized identity to the orbifolds obtained from taking the quotient of the one-holed torus by its elliptic involution, and the closed genus two surface by its hyper-elliptic involution, we obtain generalizations of the Weierstrass identities for the one-holed torus, and identities for the genus two surface, also obtained by McShane using different methods in [18], [20] and [19]. We also give an interpretation of the identity in terms of complex lengths, gaps, and the direct visual measure of the boundary.

1. Introduction

Greg McShane discovered the following striking identity in his Ph.D. thesis:

Theorem 1.1. (McShane [16]) *In a once punctured hyperbolic torus T,*

$$\sum_{\gamma} \frac{1}{1+\exp|\gamma|} = \frac{1}{2}, \tag{1}$$

where the sum extends over all simple closed geodesics on T and where $|\gamma|$ denotes the length of γ in the given hyperbolic structure.

Throughout this paper we shall always use $|\gamma|$ to denote the hyperbolic length of γ if γ is a (generalized) simple closed geodesic or a simple geodesic arc on a hyperbolic (cone-)surface. All surfaces considered in this paper are assumed to be connected and orientable.

Later McShane extended his identity to more general surfaces:

Theorem 1.2. (McShane [17]) *In a finite area hyperbolic surface M with cusps and without boundary,*

$$\sum \frac{1}{1+\exp\frac{1}{2}(|\alpha|+|\beta|)} = \frac{1}{2}, \tag{2}$$

where the sum is over all unordered pairs of simple closed geodesics α, β (where α or β might be a cusp treated as a simple closed geodesic of length 0) on M such that α, β bound with a distinguished cusp point an embedded pair of pants on M.

Note that Theorem 1.1 can be regarded as a special case of Theorem 1.2 where α, β are the same for each pair α, β.

In [18] McShane demonstrated three other closely related identities for the lengths of simple closed geodesics in each of the three Weierstrass classes on a hyperbolic torus. Recall that a hyperbolic torus T has three Weierstrass points which are the fixed points of the unique elliptic involution which maps each simple closed geodesic on T onto itself with orientation reversed, and for a Weierstrass point x on T the simple closed geodesics in the Weierstrass class which is dual to x are precisely all the simple closed geodesics on T which do not pass through x.

Theorem 1.3. (McShane [18]) *In a once punctured hyperbolic torus,*

$$\sum_{\gamma \in \mathcal{A}} \sin^{-1}\left(\frac{1}{\cosh \frac{1}{2}|\gamma|}\right) = \frac{\pi}{2}, \tag{3}$$

where the sum is over all simple closed geodesics in a Weierstrass class $\mathcal{A}$.

On the other hand, B. H. Bowditch gave an alternative proof of Theorem 1.1 using Markoff triples [6] and extended the identity in Theorem 1.1 to the case of quasi-fuchsian representations of the torus group [8] as well as to the case of hyperbolic once punctured torus bundles [7]. There are also some other generalizations along these directions, by Makoto Sakuma and his co-workers, see [2], [22].

In this paper we further generalize McShane's identity as in Theorem 1.2 to the cases of hyperbolic cone-surfaces possibly with cusps and/or geodesic boundary. (See for example [10] for basic facts on cone-manifolds.) We assume that all cone points have cone angle $\leq \pi$ (except for the one-cone torus where we allow the cone angle up to 2π). The ideas are related in spirit to those in [3] while the method of proof follows closely that of McShane's in [17]. The key points are that the assumption that all cone angles are $\leq \pi$ implies that all non-peripheral simple closed curves are essentially realizable as simple geodesics in their free (relative) homotopy classes; and that the Birman-Series result [5] on the sparsity of simple geodesics carries over to this case, in particular to simple geodesic rays emanating (normally) from a fixed boundary component. It should be noted that our result shows that the assumption of discreteness of the holonomy group is unnecessary, and that it gives identities for all hyperbolic orbifold surfaces. We also show how the result can be formulated in terms of complex lengths (Theorem 1.16) even though the situation we consider here is real. This is particularly useful, and is explored further in [26], where we show how this approach allows us to generalize McShane's identity to Schottky groups, and how the Markoff triples and analytic continuation methods adopted by Bowditch in [6] can be generalized as well. (See also [12] for related work on generalized Markoff triples.) This should also lead to generalizations of Bowditch's interpretation [7] of McShane's identity for complete hyperbolic 3-manifolds which are once punctured torus bundles over the circle to identities for the hyperbolic 3-manifolds obtained by hyperbolic Dehn surgery on such manifolds. This will be explored in future work, and should tie up nicely with the work of Sakuma in [22], and Akiyoshi-Miyachi-Sakuma in [2] and [1].

To state the most general form of our generalized McShane's identities, we need to introduce some new terminology. However, to let the reader get the flavor of the generalized identities, we first state the corresponding generalizations of Theorems 1.1 and 1.2.

Theorem 1.4. *Let T be either a hyperbolic one-cone torus where the single cone point has cone angle $\theta \in (0, 2\pi)$ or a hyperbolic one-hole torus where the single boundary geodesic has length $l > 0$. Then we have respectively*

$$\sum_{\gamma} 2\tan^{-1}\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2} + \exp|\gamma|}\right) = \frac{\theta}{2}, \tag{4}$$

$$\sum_{\gamma} 2\tanh^{-1}\left(\frac{\sinh\frac{l}{2}}{\cosh\frac{l}{2} + \exp|\gamma|}\right) = \frac{l}{2}, \tag{5}$$

where the sum in either case extends over all simple closed geodesics on T.

Theorem 1.5. *Let M be a compact hyperbolic cone-surface with a single cone point of cone angle $\theta \in (0, \pi]$ and without boundary or let M be a compact hyperbolic surface with a single boundary geodesic having length $l > 0$. Then we have respectively*

$$\sum 2\tan^{-1}\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2} + \exp\frac{|\alpha|+|\beta|}{2}}\right) = \frac{\theta}{2}, \tag{6}$$

$$\sum 2\tanh^{-1}\left(\frac{\sinh\frac{l}{2}}{\cosh\frac{l}{2} + \exp\frac{|\alpha|+|\beta|}{2}}\right) = \frac{l}{2}, \tag{7}$$

where the sum in either case extends over all unordered pairs of simple closed geodesics on M which bound with the cone point (respectively, the boundary geodesic) an embedded pair of pants.

For the purposes of this paper we make the following definition.

Definition 1.6. A **compact hyperbolic cone-surface** M is a compact (topological) surface M with hyperbolic cone structure where each boundary component is a smooth simple closed geodesic and where there are a finite number of interior points which form all the cone points and cusps. Its **geometric boundary**, denoted ΔM, is the union of all cusps, cone points and geodesic boundary components. (Note that ΔM is different from the usual topological boundary ∂M when there are cusps or cone points.) Thus a **geometric boundary component** is either a cusp, a cone point, or a boundary geodesic. The **geometric interior** of M is $M - \Delta M$.

In this paper we consider a compact hyperbolic cone-surface $M = M(\Delta_0; k, \Theta, L)$ with k cusps $C_1, C_2, \cdots, C_k$, with m cone points $P_1, P_2, \cdots, P_m$, where the cone angle of P_i is $\theta_i \in (0, \pi]$, $i = 1, 2, \cdots, m$, and with n geodesic boundary components $B_1, B_2, \cdots, B_n$, where the length of B_i is $l_i > 0$, $i = 1, 2, \cdots, n$, together with an extra *distinguished* geometric boundary component Δ_0. Thus Δ_0 is either a cusp C_0 or a cone point P_0 of cone angle $\theta_0 \in (0, \pi]$ or a geodesic boundary component B_0 of length $l_0 > 0$. Note that in the above notation $\Theta = (\theta_1, \theta_2, \cdots, \theta_m)$ and $L = (l_1, l_2, \cdots, l_n)$. We exclude the case where M is a geometric pair of pants for we have only trivial identities in that case.

We allow that some (even all) of the cone angles θ_i are equal to π, $i = 0, 1, \cdots, m$; these are often cases of particular interest. However, for clarity of exposition, quite often in proofs/statements of lemmas/theorems we shall first consider the case where all the cone angles are less than π and then point out the addenda that should be made when there are angle π cone points. The advantage of this assumption of strict inequality is that every non-trivial, non-peripheral simple closed curve on such M can be realized as a (smooth) simple closed geodesic in its free homotopy class in the geometric interior of M under the given hyperbolic cone-structure (see §4 for the proof of this statement).

We call a simple closed curve on M *peripheral* if it is freely homotopic on M to a geometric boundary component of M.

Definition 1.7. By a **generalized simple closed geodesic** on M we mean either

(i) a simple closed geodesic in the geometric interior of M; or
(ii) a degenerate simple closed geodesic which is the double of a simple geodesic arc in the geometric interior of M connecting two angle π cone points; or
(iii) a geometric boundary component, that is, a cusp or a cone point or a boundary geodesic.

In particular, generalized simple closed geodesics of the first two kinds are called **interior generalized simple closed geodesics**.

For each pair of generalized simple closed geodesics α, β which bound with Δ_0 an embedded geometric pair of pants we shall define in §3 a **gap function** $\mathrm{Gap}(\Delta_0; \alpha, \beta)$ when Δ_0 is a cone point or a boundary geodesic as well as a **normalized gap function** $\mathrm{Gap}'(\Delta_0; \alpha, \beta)$ when Δ_0 is a cusp.

Now we are in a position to state the most general (real) form of our generalization of McShane's identity.

Theorem 1.8. *Let M be a compact hyperbolic cone-surface with all cone angles in $(0, \pi]$. Then one has either*

$$\sum \mathrm{Gap}(\Delta_0; \alpha, \beta) = \frac{\theta_0}{2}, \tag{8}$$

when Δ_0 is a cone point of cone angle θ_0; or

$$\sum \mathrm{Gap}(\Delta_0; \alpha, \beta) = \frac{l_0}{2}, \tag{9}$$

when Δ_0 is a boundary geodesic of length l_0; or

$$\sum \mathrm{Gap}'(\Delta_0; \alpha, \beta) = \frac{1}{2}, \tag{10}$$

when Δ_0 is a cusp; where in each case the sum is over all pairs of generalized simple closed geodesics α, β on M which bound with Δ_0 an embedded pair of pants.

Remark 1.9.

(i) In the case of the hyperbolic one-cone torus, the theorem holds for $\theta_0 \in (0, 2\pi)$.

(ii) In the special cases where the geometric boundary ΔM is a single cone point or a single boundary geodesic Theorem 1.8 gives all the previously stated generalized identities in Theorems 1.4 and 1.5.

(iii) The cusp case (that is, Δ_0 is a cusp) is the limit case of the other cases as the cone angle θ_0 or the boundary geodesic length l_0 approaches 0, and the identity in the cusp case can indeed be derived from the first order infinitesimal of the identities of the other cases.

It is also interesting to note that McShane's Weierstrass identities can be deduced as special cases of our general Theorem 1.8 by applying the theorem to the quotient of the once punctured torus by its elliptic involution and then lifting back to the torus. Thus we have the following generalized Weierstrass identities:

Corollary 1.10. *Let T be either a hyperbolic one-cone torus where the single cone point has cone angle $\theta \in (0, 2\pi)$ or a hyperbolic one-hole torus where the single boundary geodesic has length $l > 0$. Then we have respectively*

$$\sum_{\gamma \in \mathcal{A}} \tan^{-1}\left(\frac{\cos\frac{\theta}{4}}{\sinh\frac{|\gamma|}{2}}\right) = \frac{\pi}{2}, \tag{11}$$

$$\sum_{\gamma \in \mathcal{A}} \tan^{-1}\left(\frac{\cosh\frac{l}{4}}{\sinh\frac{|\gamma|}{2}}\right) = \frac{\pi}{2}, \tag{12}$$

where the sum in either case is over all the simple closed geodesics γ in a Weierstrass class $\mathcal{A}$.

McShane's original Weierstrass identity (3) then corresponds to the case $\theta = 0$ or $l = 0$ in the above two identities, noticing that

$$\tan^{-1}\left(\frac{1}{\sinh\frac{|\gamma|}{2}}\right) = \sin^{-1}\left(\frac{1}{\cosh\frac{|\gamma|}{2}}\right).$$

As further corollaries, there are the following weaker but neater identities, each of which is obtained by summing the three McShane's Weierstrass identities in the corresponding case.

Corollary 1.11. *Let T be a hyperbolic torus whose geometric boundary is either a single cusp, a single cone point of cone angle $\theta \in (0, 2\pi)$, or a single boundary geodesic of length $l > 0$. Then we have respectively*

$$\sum_{\gamma} \tan^{-1}\left(\frac{1}{\sinh\frac{|\gamma|}{2}}\right) = \frac{3\pi}{2}, \tag{13}$$

$$\sum_{\gamma} \tan^{-1}\left(\frac{\cos\frac{\theta}{4}}{\sinh\frac{|\gamma|}{2}}\right) = \frac{3\pi}{2}, \tag{14}$$

$$\sum_{\gamma} \tan^{-1}\left(\frac{\cosh\frac{l}{4}}{\sinh\frac{|\gamma|}{2}}\right) = \frac{3\pi}{2}, \tag{15}$$

where the sum in each case is over all the simple closed geodesics γ on T.

Remark 1.12. The identity (15) was also obtained by McShane [19] using Wolpert's variation of length method. It seems likely his method can be extended to prove some of the other identities as well.

Similarly, for a genus two closed hyperbolic surface M, one can consider the (six) identities on the quotient surface M/η where η is the unique hyper-elliptic involution on M (note that M/η is a closed hyperbolic orbifold of genus 0 with six cone angle π points, and we may choose any one of these cone points to be the distinguished geometric boundary component) and re-interpret them as Weierstrass identities on the original surface M (see also McShane [20] where the Weierstrass identities were obtained directly). Combining all the six Weierstrass identities for M, we then have the following very neat identity.

Theorem 1.13. *Let M be a genus two closed hyperbolic surface. Then*

$$\sum \tan^{-1} \exp\left(-\frac{|\alpha|}{4} - \frac{|\beta|}{2}\right) = \frac{3\pi}{2}, \tag{16}$$

where the sum is over all ordered pairs (α, β) of disjoint simple closed geodesics on M such that α is separating and β is non-separating.

Remark 1.14. This is the only case that we know of where McShane's identity extends in a nice way to a closed surface.

We observe that the above identity for closed genus two surface M also extends to quasi-Fuchsian representations of $\pi_1(M)$. More precisely, let $\rho : \pi_1(M) \to \mathrm{SL}(2, \mathbf{C})$ be a quasi-Fuchsian representation, that is, $\pi \circ \rho : \pi_1(M) \to \mathrm{PSL}(2, \mathbf{C})$ is a quasi-Fuchsian representation where $\pi : \mathrm{SL}(2, \mathbf{C}) \to \mathrm{PSL}(2, \mathbf{C})$ is the projection map. For each essential simple closed curve γ, let $l_\rho(\gamma)/2 \in \mathbf{C}$ with positive real part and with imaginary part $\in (-\pi, \pi]$ be defined by

$$\mathrm{tr}\rho([\gamma]) = 2\cosh(l_\rho(\gamma)/2),$$

where $[\gamma] \in \pi_1(M)$ is the homotopy class of γ. Note that $l_\rho(\gamma)$ is also called the complex length of $\rho([\gamma])$, see for example [11].

Addendum 1.15. *For a quasi-Fuchsian representation $\rho : \pi_1(M) \to \mathrm{SL}(2, \mathbf{C})$ for the closed genus two surface M, we have*

$$\sum \tan^{-1} \exp\left(-\frac{l_\rho(\alpha)}{4} - \frac{l_\rho(\beta)}{2}\right) = \frac{3\pi}{2}, \tag{17}$$

where the sum is over all the ordered pairs $[\alpha], [\beta]$ of homotopy classes of disjoint unoriented essential simple closed curves α, β on M such that α is non-separating and β is separating.

In the statement of Theorem 1.8 we did not write down the explicit expression for the gap functions due to their "case by case" nature as can be seen in §3. The cone points and boundary geodesics as geometric boundary components seem to have different roles in the series in the generalized identities, hence making

the identities not in a unified form. This difference can, however, be removed by assigning purely imaginary length to a cone point as a geometric boundary component. More precisely, for each generalized simple closed geodesic δ, we define its **complex length** $|\delta|$ as: $|\delta| = 0$ if δ is a cusp; $|\delta| = \theta i$ if δ is a cone point of angle $\theta \in (0, \pi]$; and $|\delta| = l$ if δ is a boundary geodesic or an interior generalized simple closed geodesic of length $l > 0$. Then we can reformulate the generalized McShane's identities in Theorem 1.8 as follows.

Theorem 1.16. *Let M be a compact hyperbolic cone-surface with all cone angles in $(0, \pi]$, and let all its geometric boundary components be $\Delta_0, \Delta_1, \cdots, \Delta_N$ with complex lengths $L_0, L_1, \cdots, L_N$ respectively. Then*

$$\sum_{\alpha,\beta} 2\tanh^{-1}\left(\frac{\sinh\frac{L_0}{2}}{\cosh\frac{L_0}{2} + \exp\frac{|\alpha|+|\beta|}{2}}\right) + \sum_{j=1}^{N}\sum_{\beta}\tanh^{-1}\left(\frac{\sinh\frac{L_0}{2}\sinh\frac{L_j}{2}}{\cosh\frac{|\beta|}{2} + \cosh\frac{L_0}{2}\cosh\frac{L_j}{2}}\right) = \frac{L_0}{2}, \tag{18}$$

if Δ_0 is a cone point or a boundary geodesic; and

$$\sum_{\alpha,\beta}\frac{1}{1+\exp\frac{|\alpha|+|\beta|}{2}} + \sum_{j=1}^{N}\sum_{\beta}\frac{1}{2}\frac{\sinh\frac{L_j}{2}}{\cosh\frac{|\beta|}{2} + \cosh\frac{L_j}{2}} = \frac{1}{2}, \tag{19}$$

if Δ_0 is a cusp; where in either case the first sum is over all (unordered) pairs of generalized simple closed geodesics α, β on M which bound with Δ_0 an embedded pair of pants on M (note that one of α, β might be a geometric boundary component) and the sub-sum in the second sum is over all interior simple closed geodesics β which bounds with Δ_j and Δ_0 an embedded pair of pants on M.

Furthermore, each series in (18) and (19) converges absolutely.

Additional Remark. We were informed while writing this paper by Makoto Sakuma and Caroline Series of the recent striking results of Maryam Mirzakhani [21] where she had generalized McShane's identities to hyperbolic surfaces with boundary and used it to calculate the Weil-Petersson volumes of the corresponding moduli spaces. There is obviously an overlap of her results with ours, in particular, the identities she obtains are equivalent to ours in the case of hyperbolic surfaces with boundary (see §9 for further explanations). In fact, her expressions in terms of the log function seems particular well suited to her purpose of calculating the Weil-Petersson volumes. It also seems (as already observed by her in [21]) that her methods should extend fairly easily to cover the case of volumes of the moduli spaces of compact hyperbolic cone-surfaces with all cone angles bounded above by π, as defined and used in our context, and that the formulas she exhibited for the volumes should hold in this case as well, using the convention that a cone point of angle θ corresponds to a geometric boundary component with purely imaginary length θi .

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works of McShane [19] and Mirzakhani [21]; and Greg McShane for helpful e-mail correspondence and also for bringing our attention to [20].

2. **The organization of the rest of this paper**

The rest of this paper is organized as follows. In §3 we define the gap functions used in Theorem 1.8 for the various cases. In §4 we deal with the problem of realization of simple closed curves by geodesics, and show that the assumption that all cone angles are less than or equal to π is essential. In §5 we analyze the so-called Δ_0-geodesics, that is, the geodesics starting/emanating orthogonally from Δ_0, and determine all the gaps between all simple-normal Δ_0-geodesics. In §6 we calculate the gap function which is the width of a combined gap measured suitably. In §7 we generalize the Birman-Series theorem (which states that the point set of all complete geodesics with bounded self intersection numbers on a compact hyperbolic surface has Hausdorff dimension 1) to the case of compact hyperbolic cone-surfaces with all cone angles less than or equal to π. We prove the theorems in this paper in §8, except for Theorem 1.16, which is deferred to the last section. Finally in §9 we restate the complexified generalized McShane's identity (18) (Theorem 1.16) using two functions of complex variables and hence unify the somewhat unattractive "case-by-case" definition of the gap functions. We interpret the geometric meanings of the complexified summands in the complexified generalized McShane's identity and prove the absolute convergence of the complexified series in it by a simple use of the Birman–Series arguments in [5].

3. **Defining the Gap functions**

In this section, for a compact hyperbolic cone-surface $M = M(\Delta_0; k, \Theta, L)$ with all cone angles $\leq \pi$ we define the gap function $\mathrm{Gap}(\Delta_0; \alpha, \beta)$ (when Δ_0 is a cone point or a boundary geodesic) and the normalized gap function $\mathrm{Gap}'(\Delta_0; \alpha, \beta)$ (when Δ_0 is a cusp) where α, β are generalized simple closed geodesics on M which bound with Δ_0 a geometric pair of pants.

Throughout this paper we use $|\alpha|$ to denote the length of α when α is an interior generalized simple closed geodesic or a boundary geodesic. In particular, when α is a degenerate simple closed geodesic (that is, the double cover of a simple geodesic arc which connects two angle π cone points), its length $|\alpha|$ is defined as twice the length of the simple geodesic that it covers.

Recall that an interior generalized simple closed geodesic is either a simple closed geodesic in the geometric interior of M or a degenerate simple closed geodesic on

M which is the double cover of a simple geodesic arc which connects two angle π cone points.

Case 0. Δ_0 is a cusp.

Subcase 0.1. Both α and β are interior generalized simple closed geodesics.

In this case

$$\mathrm{Gap}'(\Delta_0;\alpha,\beta)=\frac{1}{1+\exp\frac{1}{2}(|\alpha|+|\beta|)}. \tag{20}$$

Subcase 0.2. One of α,β, say α, is a boundary geodesic and the other, β, is an interior generalized simple closed geodesic.

In this case

$$\mathrm{Gap}'(\Delta_0;\alpha,\beta)=\frac{1}{2}-\frac{1}{2}\frac{\sinh\frac{|\beta|}{2}}{\cosh\frac{|\alpha|}{2}+\cosh\frac{|\beta|}{2}}. \tag{21}$$

Subcase 0.3. One of α,β, say α, is a cone point of cone angle $\varphi\in(0,\pi]$ and the other, β, is an interior generalized simple closed geodesic.

In this case

$$\mathrm{Gap}'(\Delta_0;\alpha,\beta)=\frac{1}{2}-\frac{1}{2}\frac{\sinh\frac{|\beta|}{2}}{\cos\frac{\varphi}{2}+\cosh\frac{|\beta|}{2}}. \tag{22}$$

Subcase 0.4. One of α,β, say α, is also a cusp and the other, β, is an interior generalized simple closed geodesic.

In this case

$$\mathrm{Gap}'(\Delta_0;\alpha,\beta)=\frac{1}{2}-\frac{1}{2}\frac{\sinh\frac{|\beta|}{2}}{1+\cosh\frac{|\beta|}{2}}=\frac{1}{1+\exp\frac{1}{2}|\beta|}, \tag{23}$$

which is the common value of $\mathrm{Gap}(\Delta_0;\alpha,\beta)$ in Subcases 0.1 through 0.3 when $|\alpha|=0$.

Case 1. Δ_0 is a cone point of cone angle $\theta\in(0,\pi]$.

Subcase 1.1. Both α and β are interior generalized simple closed geodesics.

In this case

$$\mathrm{Gap}(\Delta_0;\alpha,\beta)=2\tan^{-1}\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}+\exp\frac{|\alpha|+|\beta|}{2}}\right). \tag{24}$$

Subcase 1.2. One of α,β, say α, is a boundary geodesic and the other, β, is an interior generalized simple closed geodesic.

In this case

$$\mathrm{Gap}(\Delta_0;\alpha,\beta)=\frac{\theta}{2}-\tan^{-1}\left(\frac{\sin\frac{\theta}{2}\sinh\frac{|\beta|}{2}}{\cosh\frac{|\alpha|}{2}+\cos\frac{\theta}{2}\cosh\frac{|\beta|}{2}}\right). \tag{25}$$

Subcase 1.3. One of α,β, say α, is a cone point of cone angle $\varphi\in(0,\pi]$ and the other, β, is an interior generalized simple closed geodesic.

In this case

$$\mathrm{Gap}(\Delta_0;\alpha,\beta)=\frac{\theta}{2}-\tan^{-1}\left(\frac{\sin\frac{\theta}{2}\sinh\frac{|\beta|}{2}}{\cos\frac{\varphi}{2}+\cos\frac{\theta}{2}\cosh\frac{|\beta|}{2}}\right). \tag{26}$$

Note that there is no gap when $\theta=\varphi=\pi$.

Subcase 1.4. One of α,β, say α, is a cusp and the other, β, is an interior generalized simple closed geodesic.

In this case

$$\mathrm{Gap}(\Delta_0;\alpha,\beta) = 2\tan^{-1}\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}+\exp\frac{|\beta|}{2}}\right) \tag{27}$$

$$= \frac{\theta}{2}-\tan^{-1}\left(\frac{\sin\frac{\theta}{2}\sinh\frac{|\beta|}{2}}{1+\cos\frac{\theta}{2}\cosh\frac{|\beta|}{2}}\right), \tag{28}$$

which is the common value of $\mathrm{Gap}(\Delta_0;\alpha,\beta)$ in Subcases 1.1 through 1.3 when $|\alpha|=0$.

Case 2. Δ_0 is a boundary geodesic of length $l>0$.

Subcase 2.1. Both α and β are interior generalized simple closed geodesics.

In this case

$$\mathrm{Gap}(\Delta_0;\alpha,\beta)=2\tanh^{-1}\left(\frac{\sinh\frac{l}{2}}{\cosh\frac{l}{2}+\exp\frac{|\alpha|+|\beta|}{2}}\right). \tag{29}$$

Subcase 2.2. One of α,β, say α, is a boundary geodesic and the other, β, is an interior generalized simple closed geodesic.

In this case

$$\mathrm{Gap}(\Delta_0;\alpha,\beta)=\frac{l}{2}-\tanh^{-1}\left(\frac{\sinh\frac{l}{2}\sinh\frac{|\beta|}{2}}{\cosh\frac{|\alpha|}{2}+\cosh\frac{l}{2}\cosh\frac{|\beta|}{2}}\right). \tag{30}$$

Subcase 2.3. One of α,β, say α, is a cone point of cone angle $\varphi\in(0,\pi]$ and the other, β, is an interior generalized simple closed geodesic.

In this case

$$\mathrm{Gap}(\Delta_0;\alpha,\beta)=\frac{l}{2}-\tanh^{-1}\left(\frac{\sinh\frac{l}{2}\sinh\frac{|\beta|}{2}}{\cos\frac{\varphi}{2}+\cosh\frac{l}{2}\cosh\frac{|\beta|}{2}}\right). \tag{31}$$

Subcase 2.4. One of α,β, say α, is a cusp and the other, β, is an interior generalized simple closed geodesic.

In this case

$$\mathrm{Gap}(\Delta_0;\alpha,\beta) = 2\tanh^{-1}\left(\frac{\sinh\frac{l}{2}}{\cosh\frac{l}{2}+\exp\frac{|\beta|}{2}}\right) \tag{32}$$

$$= \frac{l}{2}-\tanh^{-1}\left(\frac{\sinh\frac{l}{2}\sinh\frac{|\beta|}{2}}{1+\cosh\frac{l}{2}\cosh\frac{|\beta|}{2}}\right), \tag{33}$$

which is the common value of $\mathrm{Gap}(\Delta_0;\alpha,\beta)$ in Subcases 2.1 through 2.3 when $|\alpha|=0$.

4. Realizing simple curves by geodesics on hyperbolic cone-surfaces

In this section we consider the problem of realizing essential simple curves in their free (relative) homotopy classes by geodesics on a compact hyperbolic cone-surface M with all cone angles smaller than π. We show that each essential simple closed curve in the geometric interior of M can be realized uniquely in its free homotopy class (where the homotopy takes place in the geometric interior of M) as either a geometric boundary component or a simple closed geodesic in the geometric interior of M. We also show that each essential simple arc which connects geometric boundary components of M can be realized uniquely in its free relative homotopy class (where the homotopy takes place in the geometric interior of M and the endpoints slide on the same geometric boundary components) as a simple geodesic arc which is normal to the geometric boundary components involved. We also make addenda for the cases when there are angle π cone points.

Theorem 4.1. *Let M be a compact hyperbolic cone-surface with all cone angles less than π.*

(i) If c is an essential non-peripheral simple closed curve in the geometric interior of M, then there is a unique simple closed geodesic in the free homotopy class of c in the geometric interior of M.

(ii) If c is an essential simple arc which connects geometric boundary components, then there is a unique simple normal geodesic arc in the free relative homotopy class of c in the geometric interior of M with endpoints varying on the respective geometric boundary components.

Addendum 4.2. *If in addition M has some cone angles equal to π, then*

(i) *in Theorem 4.1(i), if the simple closed curve c bounds with two angle π cone points an embedded pair of pants, then the geodesic realization for c is the double cover of the simple geodesic arc which connects these two angle π cone points and is homotopic (relative to boundary) to a simple arc lying wholly in the pair of pants;*

(ii) *in Theorem 4.1(ii), if the simple arc c connects a geometric boundary component Δ to itself and bounds together with Δ and an angle π cone point P an embedded cylinder then the geodesic realization for c is the double cover of the normal simple geodesic arc which connects Δ to P and is homotopic (relative to boundary) to a simple arc lying wholly in the cylinder.*

The simple geodesic in Theorem 4.1 and Addendum 4.2 is called the geodesic realization of the given simple curve in the respective homotopy class.

The proof is a well-known use of the Arzela-Ascoli Theorem as used in [9] with slight modifications.

Proof: (i) Suppose c is an essential non-peripheral simple closed curve in the geometric interior of M, parameterized on $[0,1]$ with constant speed. Let the length of c be $|c| > 0$. Then for each cusp C_i, there is an embedded neighborhood $N(C_i)$ of C_i on M, bounded by a horocycle, such that each non-peripheral simple closed curve c' in the geometric interior of M with length $\leq |c|$ cannot enter $N(C_i)$; for otherwise c' would be either peripheral or of infinite length. Now let M_0 be M with all the chosen horocycle neighborhoods $N(C_i)$ removed. Then M_0 is a compact metric subspace of M with the induced hyperbolic metric. Now choose a sequence of simple closed curves $\{c_k\}_1^\infty$, where each c_k is parameterized on $[0,1]$ with constant speed, in the free homotopy class of c (where the homotopy takes place in the geometric interior of M) such that their lengths $\leq |c|$ and are decreasing with limit the infimum of the lengths of the simple closed curves in the free homotopy class of c. Then by the Arzela-Ascoli Theorem (c.f. [9] Theorem A.19, page 429) there is a subsequence of $\{c_k\}_1^\infty$, assumed to be $\{c_k\}_1^\infty$ itself, such that it converges uniformly to a closed curve γ in M_0. It is clear that γ is a geodesic since it is locally minimizing. Note that γ is away from cusps by the choice of $\{c_k\}_1^\infty$. We claim that γ cannot pass through any cone point. For otherwise, suppose γ passes through a cone point P. Then for sufficiently large k, c_k can be modified in the free homotopy class of c to have length smaller than $|\gamma|$ (since the cone point has cone angle smaller than π), which is a contradiction. Thus γ must be a closed geodesic in the geometric interior of M. The uniqueness and simplicity of γ can be proved by an easy argument since there are no bi-gons in the hyperbolic plane.

(ii) For an essential simple arc c in the geometric interior of M which connects geometric boundary components, the proof of case (i) applies without modifications when none of the involved geometric boundary components is a cusp. Now suppose at least one of the involved geometric boundary components is a cusp. For definiteness let us assume that c connects cusps C_1 to C_2. Remove suitable horocycle neighborhoods $N(C_1)$ and $N(C_2)$ respectively for C_1 and C_2 where the two horocycles are H_1 and H_2 respectively. Choose a simple arc c_0 in $M - N(C_1) \cup N(C_2)$ which goes along c and connects H_1 to H_2. Let the length of c_0 be $|c_0| > 0$. Now for all other cusps C_i, there is a horocycle neighborhood $N(C_i)$ of C_i on M such that each non-peripheral simple closed curve c' in the geometric interior of M with

length $\leq |c_0|$ cannot enter $N(C_i)$. Again let M_0 be M with all the chosen horocycle neighborhoods $N(C_i)$ removed. By the same argument as in (i) we have a shortest simple geodesic realization γ_0 in the free relative homotopy class of c_0 in M_0 and c_0 does not pass through any cone point. Hence γ_0 must be perpendicular to both H_1 and H_2 at its endpoints. Thus γ_0 can be extended to a geodesic arc connecting C_1 to C_2. Again simplicity and uniqueness can be proved easily. □

The addendum can be verified easily since the realizations as degenerate simple geodesics in the respective cases are already known.

Remark 4.3. We make a remark that the following fact, whose proof is easy and hence omitted, is implicitly used through out this paper: On a hyperbolic cone-surface for each cone point P with angle less than π there is a cone region $N(P)$, bounded by a suitable circle centered at P, such that if a geodesic γ goes into $N(P)$ then either γ will go directly to the cone point P (hence perpendicular to all the circles centered at P) or γ will develop a self-intersection in $N(P)$. The analogous fact for a cusp is used in [5], [13] and [17].

5. **Gaps between simple-normal Δ_0-geodesics**

Definition 5.1. A **Δ_0-geodesic** on M is an *oriented* geodesic ray which starts from Δ_0 (and is perpendicular to it if Δ_0 is a boundary geodesic) and is fully developed, that is, it develops forever until it terminates at a geometric boundary component. We denote by $\mathcal{G}(\Delta_0)$ (or just $\mathcal{G}$) the set of Δ_0-geodesics.

A Δ_0-geodesic is either non-simple or simple. It is regarded as **non-simple** if and only if it intersects itself transversely at an interior point (a cone point is not treated as an interior point) or at a point on a boundary geodesic. We shall see later that somewhat surprisingly, in some sense, the set of non-simple Δ_0-geodesics is easier to analyze than the set of simple Δ_0-geodesics.

A simple Δ_0-geodesic is either normal or not-normal in the following sense:
A simple Δ_0-geodesic is **normal** if when fully developed either it never intersects any boundary geodesic or it intersects (hence terminates at) a boundary geodesic perpendicularly. Note that a simple-normal Δ_0-geodesic may terminate at a cusp or a cone point. Thus a simple Δ_0-geodesic is **not-normal** if and only if it intersects a boundary geodesic (which might be Δ_0 itself) obliquely.

We shall analyze the structure of all non-simple and simple-not-normal Δ_0-geodesics and show that they form gaps between simple-normal Δ_0-geodesics. Furthermore, the naturally measured widths of the suitably combined gaps are given by the Gap functions defined before in §3.

Note that McShane [17] analyzes directly all simple Δ_0-geodesics (there are no simple-not-normal Δ_0-geodesics in his case since there are no geodesic boundary componenets). Our analysis of the structure of Δ_0-geodesics is a bit different from and actually simpler than that of McShane's. We shall analyze all non-simple and simple-not-normal Δ_0-geodesics and show that they arise in the nice ways we expect.

First we parameterize all the Δ_0-geodesics and define the widths for gaps between simple-normal Δ_0-geodesics.

If Δ_0 is a cusp let $\mathcal{H}$ be a suitably chosen small horocycle as in McShane [17], see also [13]. If Δ_0 is a cone point let $\mathcal{H}$ be a suitably chosen small circle centered at Δ_0. Let $\mathcal{H}$ be Δ_0 itself if Δ_0 is a boundary geodesic.

Then each Δ_0-geodesic has a unique first intersection point with $\mathcal{H}$, which is the starting point when Δ_0 is a boundary geodesic. Note that the Δ_0-geodesics intersect $\mathcal{H}$ orthogonally at their first intersection points. Thus $\mathcal{G}$ can be naturally identified with $\mathcal{H}$, with the induced topology and measure. Let $\mathcal{H}_{\tt ns}$, $\mathcal{H}_{\tt sn}$, $\mathcal{H}_{\tt snn}$ be the point sets of the first intersections of $\mathcal{H}$ with respectively all non-simple, all simple-normal, all simple-not-normal Δ_0-geodesics.

Proposition 5.2. *The set $\mathcal{H}_{\tt ns} \cup \mathcal{H}_{\tt snn}$ is an open subset of $\mathcal{H}$ and hence $\mathcal{H}_{\tt sn}$ is a closed subset of $\mathcal{H}$.*

Proof. It is easy to see that the condition that either self-intersecting or ending obliquely at a boundary component is an open condition. $\square$

For the open subset $\mathcal{H}_{\tt ns} \cup \mathcal{H}_{\tt snn}$ of $\mathcal{H}$, we determine its structure by determining its maximal open intervals (which are the gaps we are looking for). By a generalized Birman–Series Theorem (see §7), the subset $\mathcal{H}_{\tt sn}$ of $\mathcal{H}$ has Hausdorff dimension 0, and hence Lebesgue measure 0. Therefore the open subset $\mathcal{H}_{\tt ns} \cup \mathcal{H}_{\tt snn}$ of $\mathcal{H}$ has full measure, and our generalized McShane's identities (8)-(10) follow immediately.

Definition 5.3. A $[\Delta_0, \Delta_0]$-geodesic, γ, is an (oriented) Δ_0-geodesic which terminates at Δ_0 perpendicularly. (With the orientation one can refer to its starting point and ending point.) Hence the same geodesic with reversed orientation (hence with the starting and ending points interchanged) is also a $[\Delta_0, \Delta_0]$-geodesic, denoted by $-\gamma$.

We say that a $[\Delta_0, \Delta_0]$-geodesic γ is a **degenerate simple** $[\Delta_0, \Delta_0]$**-geodesic** if Δ_0 is not a π cone point, and γ is the double cover of a simple geodesic arc which connects Δ_0 to an angle π cone point, that is, γ reaches the angle π cone point along the simple geodesic arc and goes back to Δ_0 along the same arc. Note that in this case $\gamma = -\gamma$.

We show that each non-degenerate simple $[\Delta_0, \Delta_0]$-geodesic γ determines two maximal open intervals of $\mathcal{H}_{\tt ns} \cup \mathcal{H}_{\tt snn}$ as follows. (Their union is the *main gap*, defined later, determined by γ.)

Consider the configuration $\gamma \cup \mathcal{H}$. Assume γ is non-degenerate and let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the two sub-arcs with endpoints inclusive that γ divides $\mathcal{H}$ into. Note that γ

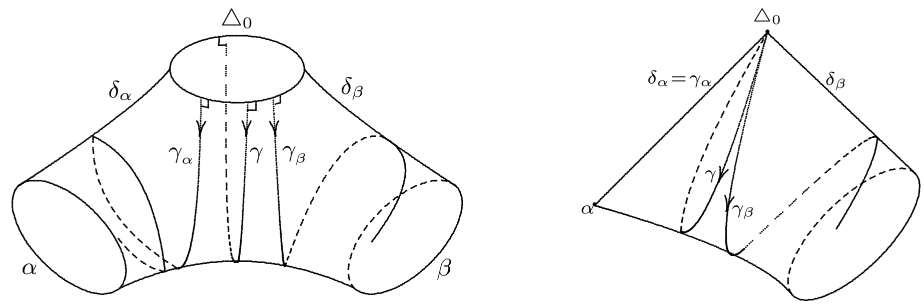

FIGURE 1.

intersects $\mathcal{H}$ twice (if $\mathcal{H}$ is taken to be a suitably small circle about Δ_0 when Δ_0 is a cone point). Let γ_0 be the sub-arc of γ between the two intersection points. Thus we have two simple closed curves $\mathcal{H}_1 \cup \gamma_0$ and $\mathcal{H}_2 \cup \gamma_0$ on M. Their geodesic realizations are disjoint generalized simple closed geodesics, denoted α, β respectively (except when M is a hyperbolic torus with a single geometric boundary component, in which case $\alpha = \beta$). Note that α, β bound with Δ_0 an embedded geometric pair of pants, denoted $\mathcal{P}(\gamma)$, on M.

Let δ_α be the simple Δ_0-geodesic arc in $\mathcal{P}(\gamma)$ which terminates at α and is normal to α. Similarly, let δ_β be the simple Δ_0-geodesic arc in $\mathcal{P}(\gamma)$ which terminates at β and is normal to β. Let $[\alpha, \beta]$ be the simple geodesic arc in $\mathcal{P}(\gamma)$ which connects α and β and is normal to them. See Figure 1.

Cutting $\mathcal{P}(\gamma)$ along $\delta_\alpha, \delta_\beta$ and $[\alpha, \beta]$ one obtains two pieces; let the one which contains the initial part of γ be denoted $\mathcal{P}^+(\gamma)$. There are two simple Δ_0-geodesics, γ_α and γ_β, in $\mathcal{P}(\gamma)$ such that they are asymptotic to α and β respectively, and such that their initial parts are contained in $\mathcal{P}^+(\gamma)$. See Figure 1.

Lemma 5.4. *Each Δ_0-geodesic whose initial part lies in $\mathcal{P}^+(\gamma)$ between γ_α and γ or between γ and γ_β is non-simple or simple-not-normal.*

The union of these two gaps between simple-normal Δ_0-geodesics formed by non-simple and simple-not-normal Δ_0-geodesics is called the **main gap** determined by γ.

This lemma can be proved easily using a suitable model of the hyperbolic plane; see [27] for details. The idea is that a Δ_0-geodesic ray whose initial part lies in $\mathcal{P}^+(\gamma)$ between γ_α and γ will not intersect γ_α or γ directly, so it must come back to intersect for first time either itself or Δ_0, hence is either non-simple or simple but not-normal (that is, intersecting Δ_0 obliquely). More precisely, if Δ_0 is a cusp or a cone point all the Δ_0-geodesics in the lemma are non-simple, while if Δ_0 is a boundary geodesic then there is a (critical) Δ_0-geodesic, ρ_γ, whose initial part lies in $\mathcal{P}^+(\gamma)$ between γ_α and γ such that ρ_γ is non-simple and its only self-intersection is at its starting point on Δ_0 (and hence terminates there) and it has the property that each Δ_0-geodesic whose initial part lies in $\mathcal{P}^+(\gamma)$ between γ_α and ρ_γ is non-simple, while each Δ_0-geodesic whose initial part lies in $\mathcal{P}^+(\gamma)$ between ρ_γ and

γ is simple-not-normal terminating at Δ_0. There is a similar dichotomy for the Δ_0-geodesics whose initial parts lie in $\mathcal{P}^+(\gamma)$ between γ and γ_β.

Now suppose one of α, β, say α, is a boundary geodesic. Then there are two simple Δ_0-geodesics in $\mathcal{P}(\gamma)$ which are asymptotes to α. They are γ_α and $(-\gamma)_\alpha$.

The following lemma tells us that there is an **extra gap** determined by γ in $\mathcal{P}^+(\gamma)$ between simple-normal Δ_0-geodesics formed by simple-not-normal Δ_0-geodesics.

Lemma 5.5. *Each Δ_0-geodesic whose initial part lies in $\mathcal{P}^+(\gamma)$ between δ_α and γ_α is simple-not-normal.*

This is almost self-evident from the geometry of the pair of pants $P(\gamma)$, and is similar to the proof of the previous lemma; see [27] for details.

Note that there is a similar and symmetric picture for the Δ_0-geodesics whose initial parts lie in $\mathcal{P}^-(\gamma)$.

Hence (for non-degenerate γ) in the geometric pair of pants $\mathcal{P}(\gamma)$, which is the same as $\mathcal{P}(-\gamma)$, if none of α, β is a boundary geodesic then there are two main gaps determined by γ and $-\gamma$ respectively; if (exactly) one of α, β is a boundary geodesic then there are two extra gaps determined by γ and $-\gamma$.

The case of a degenerate simple $[\Delta_0, \Delta_0]$-geodesic γ is handled in a similar way. Recall that γ is the double cover of a Δ_0-geodesic arc δ from Δ_0 to an angle π cone point α. Then there is a simple closed curve β', which is the boundary of a suitable regular neighborhood of $\Delta_0 \cup \delta$ on M, such that β' bounds with Δ_0 and α an embedded (topological) pair of pants. If Δ_0 is not itself an angle π cone point, then β' can be realized as an interior generalized simple closed geodesic β which bounds with Δ_0 and α an embedded pair of pants $\mathcal{H}(\Delta_0, \alpha, \beta)$ on M and we can carry out the analysis as above with suitable modifications. In this case γ determines no gaps if Δ_0 is itself an angle π cone point. If Δ_0 is not itself an angle π cone point then there are two main gaps, between γ and each of the two Δ_0-geodesics which are asymptotic to β in $\mathcal{H}(\Delta_0, \alpha, \beta)$. We say that one of the two main gaps is determined by γ and the other by $-\gamma$ although $\gamma = -\gamma$ in this case.

Definition 5.6. The **width** of an open subinterval $\mathcal{H}'$ of $\mathcal{H}$ is defined respectively as:

(i) Δ_0 is a cusp: the normalized parabolic measure, that is, the ratio of the Euclidean length of $\mathcal{H}'$ to the Euclidean length of $\mathcal{H}$;
(ii) Δ_0 is a cone point: the elliptic measure, that is, the angle (measured in radians) that $\mathcal{H}'$ subtends with respect to the cone point Δ_0;
(iii) Δ_0 is a boundary geodesic: the hyperbolic measure, that is, the hyperbolic length of $\mathcal{H}'$ (recall that in this case $\mathcal{H}$ is the same as the distinguished boundary geodesic Δ_0).

Definition 5.7. The **combined gap** between simple-normal Δ_0-geodesics determined by γ is the union of the main gap and the extra gap (if there is any) determined by γ. The **gap function** $\mathrm{Gap}(\Delta_0; \alpha, \beta)$ when Δ_0 is a cone point or

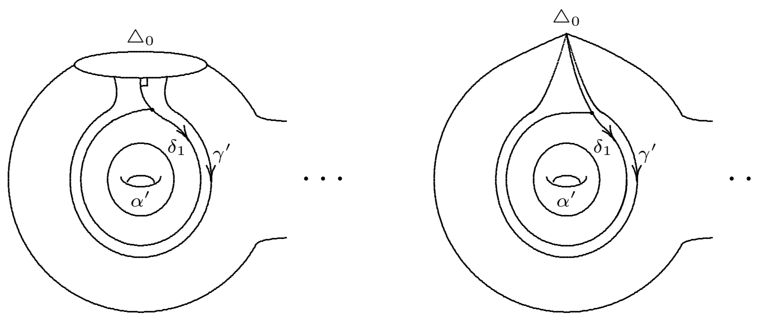

FIGURE 2.

boundary geodesic or the **normalized gap function** $\mathrm{Gap}'(\Delta_0; \alpha, \beta)$ when Δ_0 is a cusp is defined as the total width of the combined gap determined by γ, which is by symmetry the same as the total width of the combined gap determined by $-\gamma$.

We shall calculate the the gap functions in §6.

On the other hand, the following key lemma shows that the non-simple and simple-not-normal Δ_0-geodesics obtained above are *all* the non-simple and simple-not-normal Δ_0-geodesics.

Lemma 5.8. *Each non-simple or simple-not-normal Δ_0-geodesic lies in a main gap or an extra gap determined by some $[\Delta_0, \Delta_0]$-geodesic γ.*

Proof: First let δ be a non-simple Δ_0-geodesic, with its first self-intersection point Q, where Q lies in the geometric interior of M or in Δ_0 when Δ_0 is a boundary geodesic. Let δ_1 be the part of δ from starting point to Q; note that δ_1 has the shape of a lasso. Then in the boundary of a suitable regular neighborhood of δ_1 there is a simple arc γ' which connects Δ_0 to itself and is disjoint from δ_1 (except at Δ_0 when Δ_0 is a cone point); there is also a simple closed curve α' which is freely homotopic to the loop part of δ_1. See Figure 2. Let γ, α be the generalized simple closed geodesics on M which realize γ', α' in their respective free (relative) homotopy classes in the geometric interior of M. An easy geometric argument shows that α is disjoint from δ_1 and that γ is also disjoint from δ_1 except at Δ_0 when Δ_0 is a cone point or a cusp. Furthermore, γ and α cobound (together with Δ_0 when Δ_0 is a boundary geodesic) an embedded cylinder which contains δ_1. Hence the point in $\mathcal{H}$ which corresponds to the Δ_0-geodesic δ lies in the main gap determined by γ. See Figure 3

Next let δ be a simple-not-normal Δ_0-geodesic which terminates at Δ_0 itself; in this case Δ_0 is a boundary geodesic and $\mathcal{H}$ is Δ_0 itself. Then the boundary of a suitably chosen regular neighborhood of $\delta \cup \mathcal{H}$ consist of two disjoint simple closed curves in the geometric interior of M. Let their geodesic realizations be (disjoint) generalized simple closed geodesics α and β. Then α, β bound with Δ_0 an embedded pair of pants which contains δ in a main gap determined by the $[\Delta_0, \Delta_0]$-geodesic γ which is the geodesic realization of δ in its free relative homotopy class.

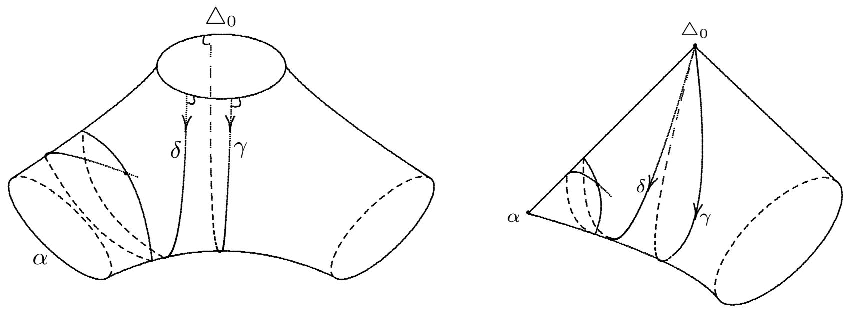

FIGURE 3.

Finally let δ be a simple-not-normal Δ_0-geodesic which terminates at a boundary geodesic Δ_1 which is different from Δ_0. The boundary of suitably chosen regular neighborhood of $\delta \cup \Delta_1$ on M is a simple arc connecting Δ_0 to itself and is disjoint from δ. Its geodesic realization is a $[\Delta_0, \Delta_0]$-geodesic, γ, which is disjoint from δ. Now Δ_1, γ bound with Δ_0 an embedded cylinder which contains δ. Hence δ lies in the extra gap determined by γ or $-\gamma$. $\square$

6. **Calculating the gap functions**

In this section we calculate the gap function $\mathrm{Gap}(\Delta_0; \alpha, \beta)$ when Δ_0 is a cone point or a boundary geodesic, it is the width of the combined gap determined by a simple $[\Delta_0, \Delta_0]$-geodesic γ on M.

Recall that α, β are the generalized simple closed geodesics determined by γ and $\mathcal{P}(\gamma)$ is the geometric pair of pants that α, β bound with Δ_0 on M.

Case 1. Δ_0 is a cone point of cone angle $\theta \in (0, \pi]$.

In this case the width of the main gap determined by γ is the angle between γ_α and γ_β.

Let x be the angle between δ_α and γ_α and let y be the angle between δ_β and γ_β.

Subcase 1.1. Both α and β are interior generalized simple closed curves.

In this case the width of the combined gap determined by γ is the angle between γ_α and γ_β and is equal to $\frac{\theta}{2} - (x + y)$.

By a formula in Fenchel [11] VI.3.2 (line 10, page 87),

$$\sinh|\delta_\alpha| = \frac{\cosh\frac{|\beta|}{2} + \cos\frac{\theta}{2}\cosh\frac{|\alpha|}{2}}{\sin\frac{\theta}{2}\sinh\frac{|\alpha|}{2}}, \tag{34}$$

$$\sinh|\delta_\beta| = \frac{\cosh\frac{|\alpha|}{2} + \cos\frac{\theta}{2}\cosh\frac{|\beta|}{2}}{\sin\frac{\theta}{2}\sinh\frac{|\beta|}{2}}. \tag{35}$$

Hence

$$\tan x = \frac{1}{\sinh|\delta_\alpha|} = \frac{\sin\frac{\theta}{2}\sinh\frac{|\alpha|}{2}}{\cosh\frac{|\beta|}{2} + \cos\frac{\theta}{2}\cosh\frac{|\alpha|}{2}}, \tag{36}$$

$$\tan y = \frac{1}{\sinh|\delta_\beta|} = \frac{\sin\frac{\theta}{2}\sinh\frac{|\beta|}{2}}{\cosh\frac{|\alpha|}{2} + \cos\frac{\theta}{2}\cosh\frac{|\beta|}{2}}. \tag{37}$$

From these one can derive that

$$\tan(x+y) = \frac{\sin\frac{\theta}{2}\sinh\frac{|\alpha|+|\beta|}{2}}{1+\cos\frac{\theta}{2}\cosh\frac{|\alpha|+|\beta|}{2}} \tag{38}$$

and hence that

$$\tan\frac{x+y}{2} = \tan\frac{\theta}{4}\tanh\frac{|\alpha|+|\beta|}{4}. \tag{39}$$

Thus

$$\tan\left(\frac{\theta}{4} - \frac{x+y}{2}\right) = \frac{\tan\frac{\theta}{4}\left(1-\tanh\frac{|\alpha|+|\beta|}{4}\right)}{1+\tan^2\frac{\theta}{4}\ \tanh\frac{|\alpha|+|\beta|}{4}} \tag{40}$$

$$= \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2} + \exp\frac{|\alpha|+|\beta|}{2}}. \tag{41}$$

Hence in this case we have

$$\begin{aligned}\mathrm{Gap}(\Delta_0;\alpha,\beta) &= \frac{\theta}{2} - (x+y)\\ &= 2\tan^{-1}\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2} + \exp\frac{|\alpha|+|\beta|}{2}}\right).\end{aligned}$$

Subcase 1.2. α is a boundary geodesic and β is an interior generalized simple closed geodesic.

In this case the width of the combined gap determined by γ is the angle between δ_α and γ_β and is equal to $\frac{\theta}{2} - y$. Hence by (37) we have

$$\mathrm{Gap}(\Delta_0;\alpha,\beta) = \frac{\theta}{2} - \tan^{-1}\left(\frac{\sin\frac{\theta}{2}\sinh\frac{|\beta|}{2}}{\cosh\frac{|\alpha|}{2} + \cos\frac{\theta}{2}\cosh\frac{|\beta|}{2}}\right). \tag{42}$$

Subcase 1.3. α is a cone point of cone angle $\varphi \in (0,\pi]$ and β is an interior generalized simple closed geodesic.

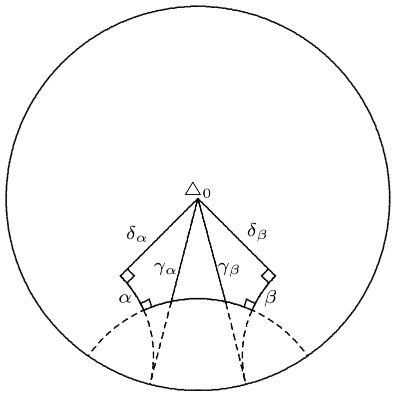

FIGURE 4. Subcase 1.1

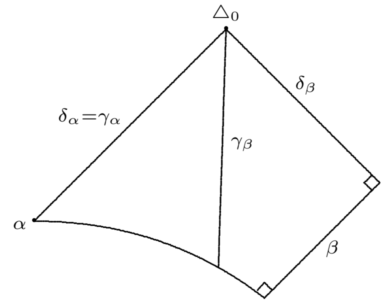

FIGURE 5. Subcase 1.3

Note that in this case γ_α coincides with δ_α and hence $x = 0$. Therefore the width of the combined gap determined by γ is the angle between δ_α and γ_β and is equal to $\frac{\theta}{2} - y$.

Now by a formula in Fenchel [11] VI.3.3 (line 13, page 88),

$$\sinh|\delta_\beta| = \frac{\cos\frac{\varphi}{2} + \cos\frac{\theta}{2}\cosh\frac{|\beta|}{2}}{\sin\frac{\theta}{2}\sinh\frac{|\beta|}{2}}. \tag{43}$$

Hence

$$\tan y = \frac{1}{\sinh|\delta_\beta|} = \frac{\sin\frac{\theta}{2}\sinh\frac{|\beta|}{2}}{\cos\frac{\varphi}{2} + \cos\frac{\theta}{2}\cosh\frac{|\beta|}{2}}. \tag{44}$$

Thus in this case we have

$$\mathrm{Gap}(\Delta_0; \alpha, \beta) = \frac{\theta}{2} - \tan^{-1}\left(\frac{\sin\frac{\theta}{2}\sinh\frac{|\beta|}{2}}{\cos\frac{\varphi}{2} + \cos\frac{\theta}{2}\cosh\frac{|\beta|}{2}}\right). \tag{45}$$

Case 2. Δ_0 is a boundary geodesic of length $l > 0$.

In this case the width of the main gap determined by γ is the distance between γ_α and γ_β along Δ_0.

Let x be the distance between δ_α and γ_α along Δ_0 and let y be the distance between δ_β and γ_β along Δ_0.

We shall see that all calculations in this case are parallel to those in Case 1.

Subcase 2.1. Both α and β are interior generalized simple closed curves.

In this case the width of the combined gap determined by γ is the distance between γ_α and γ_β along Δ_0 and is equal to $\frac{l}{2} - (x + y)$.

By the cosine rule for right angled hexagons on the hyperbolic plane (c.f. Fenchel [11] VI.3.1, page 86, or Beardon [4] Theorem 7.19.2, page 161),

$$\cosh|\delta_\alpha| = \frac{\cosh\frac{|\beta|}{2} + \cosh\frac{l}{2}\cosh\frac{|\alpha|}{2}}{\cosh\frac{l}{2}\sinh\frac{|\alpha|}{2}}, \tag{46}$$

$$\cosh|\delta_\beta| = \frac{\cosh\frac{|\alpha|}{2} + \cosh\frac{l}{2}\cosh\frac{|\beta|}{2}}{\sinh\frac{l}{2}\sinh\frac{|\beta|}{2}}. \tag{47}$$

Hence

$$\tanh x = \frac{1}{\cosh|\delta_\alpha|} = \frac{\sinh\frac{l}{2}\sinh\frac{|\alpha|}{2}}{\cosh\frac{|\beta|}{2} + \cosh\frac{l}{2}\cosh\frac{|\alpha|}{2}}, \tag{48}$$

$$\tanh y = \frac{1}{\cosh|\delta_\beta|} = \frac{\sinh\frac{l}{2}\sinh\frac{|\beta|}{2}}{\cosh\frac{|\alpha|}{2} + \cosh\frac{l}{2}\cosh\frac{|\beta|}{2}}. \tag{49}$$

From these one can derive that

$$\tanh(x+y) = \frac{\sinh\frac{l}{2}\sinh\frac{|\alpha|+|\beta|}{2}}{1+\cosh\frac{l}{2}\cosh\frac{|\alpha|+|\beta|}{2}} \tag{50}$$

and hence that

$$\tanh\frac{x+y}{2} = \tanh\frac{l}{4}\tanh\frac{|\alpha|+|\beta|}{4}. \tag{51}$$

Thus

$$\begin{aligned}
\tanh\left(\frac{l}{4}-\frac{x+y}{2}\right) &= \frac{\tanh\frac{l}{4}\left(1-\tanh\frac{|\alpha|+|\beta|}{4}\right)}{1-\tanh^2\frac{l}{4}\,\tanh\frac{|\alpha|+|\beta|}{4}} && (52)\\
&= \frac{\sinh\frac{l}{2}}{\cosh\frac{l}{2}+\exp\frac{|\alpha|+|\beta|}{2}}. && (53)
\end{aligned}$$

Hence in this case we have

$$\begin{aligned}
\mathrm{Gap}(\Delta_0;\alpha,\beta) &= \frac{l}{2}-(x+y)\\
&= 2\tanh^{-1}\left(\frac{\sinh\frac{l}{2}}{\cosh\frac{l}{2}+\exp\frac{|\alpha|+|\beta|}{2}}\right).
\end{aligned}$$

Subcase 2.2. α is a boundary geodesic and β is an interior generalized simple closed geodesic.

In this case the width of the combined gap determined by γ is the distance between δ_α and γ_β along Δ_0 and is equal to $\frac{l}{2}-y$. Hence by (49) we have

$$\mathrm{Gap}(\Delta_0;\alpha,\beta) = \frac{l}{2} - \tanh^{-1}\left(\frac{\sinh\frac{l}{2}\sinh\frac{|\beta|}{2}}{\cosh\frac{|\alpha|}{2}+\cosh\frac{l}{2}\cosh\frac{|\beta|}{2}}\right). \tag{54}$$

Subcase 2.3. α is a cone point of cone angle $\varphi\in(0,\pi]$ and β is an interior generalized simple closed geodesic.

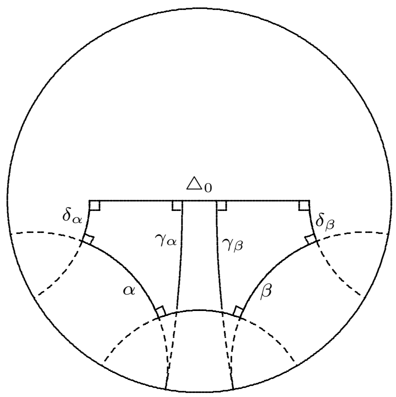

FIGURE 6. Subcase 2.1

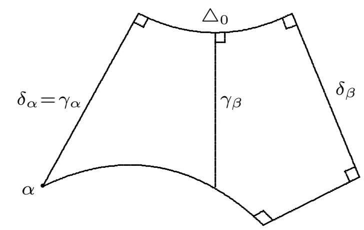

FIGURE 7. Subcase 2.3

Note that in this case γ_α coincides with δ_α and hence $x = 0$. Hence the width of the combined gap determined by γ is the distance between δ_α and γ_β along Δ_0 and is equal to $\frac{l}{2} - y$.

Now by a formula in Fenchel [11] VI.3.2 (line 8, page 87),

$$\cosh|\delta_\beta| = \frac{\cos\frac{\varphi}{2} + \cosh\frac{l}{2}\cosh\frac{|\beta|}{2}}{\sinh\frac{l}{2}\sinh\frac{|\beta|}{2}}. \tag{55}$$

Hence

$$\tanh y = \frac{1}{\cosh|\delta_\beta|} = \frac{\sinh\frac{l}{2}\sinh\frac{|\beta|}{2}}{\cos\frac{\varphi}{2} + \cosh\frac{l}{2}\cosh\frac{|\beta|}{2}}. \tag{56}$$

Thus in this case we have

$$\mathrm{Gap}(\Delta_0; \alpha, \beta) = \frac{l}{2} - \tanh^{-1}\left(\frac{\sinh\frac{l}{2}\sinh\frac{|\beta|}{2}}{\cos\frac{\varphi}{2} + \cosh\frac{l}{2}\cosh\frac{|\beta|}{2}}\right). \tag{57}$$

Remark 6.1. We remark that the formulas in Case 0 for the normalized width $\mathrm{Gap}'(\Delta_0; \alpha, \beta)$ when Δ_0 is a cusp can be derived by similar (and simpler) calculations or by considering the first order infinitesimal terms of those formulas with respect to θ in Case 1 or with respect to l in Case 2. Hence all derivations in Case 0 are omitted.

7. **Generalization of the Birman–Series Theorem**

The celebrated Birman–Series Theorem [5] in its simplest form states that complete simple geodesics on a closed hyperbolic surface are sparsely distributed.

More precisely, let M be a hyperbolic surface possibly with boundary such that M is either compact or obtained from a compact surface by removing a finite set of points which form the cusps and such that each boundary component of M is a simple closed geodesic. A geodesic on M is said to be *complete* if it is either closed and smooth, or open and of infinite length in both directions. Hence a complete geodesic never intersects ∂M. Let G_k be the family of complete geodesics on M which have at most k, counted with multiplicity, transversal self-intersections, $k \geq 0$. Then the main result in [5] is:

Theorem 7.1. *For each $k \geq 0$, the point set S_k which is the union of all geodesics, as point sets, in G_k is nowhere dense and has Hausdorff dimension one.*

In this section we show that this theorem extends to the case when M is a compact hyperbolic cone-surface with geometric boundary where each cone point has cone angle in $(0, \pi]$, with complete geodesics replaced by complete-normal ones. This is the set of geodesics which are either complete, or intersect the boundary perpendicularly.

Theorem 7.2. *Let M be a compact hyperbolic cone-surface with geometric boundary where each cone point has cone angle in $(0, \pi]$, and let G_k be the family of complete-normal geodesics on M which have at most k transversal self-intersections, $k \geq 0$. Then, for each $k \geq 0$, the point set S_k which is the union of all geodesics, as point sets, in G_k is nowhere dense and has Hausdorff dimension one.*

The proof of this generalization is is essentially the same as that of the original Birman–Series theorem given in [5]. Hence for simplicity we shall only sketch the proof of the theorem for the case $k = 0$, that is, for simple complete-normal geodesics; the reader is referred to [5] for omitted details.

We only need to consider the case where M has no geodesic boundary components; for if M has nonempty geodesic boundary we can replace M by the double of M along its geodesic boundary. We also assume for clarity that each cone point of M has cone angle less than π. We decompose the set G_0 into finitely many subsets and prove the conclusion for each such subset. For the subset of simple complete geodesics on M, that is, the geodesics which never start from or terminate at cusps or cone points, the proof is the same as that in [5] with little modification (which can be seen from the sketch below). For the subset of simple normal geodesics which connect a given cusp or cone point to another (possibly the same) given cusp or cone point, it is easy to see that in this subset each such geodesic is isolated in suitable neighborhoods of its endpoints and hence the conclusion follows. Thus it remains to prove the conclusion for the subset of simple complete-normal geodesics which starts from a given cusp or cone point P and never terminates at any geometric boundary component.

One can cut M along normal geodesics connecting cusps or cone points to form a (convex) fundamental polygon R for M in the hyperbolic plane. Let $A = \{a_1, a_2, \cdots, a_m\}$ denote the ordered set of vertices and oriented sides of R with anti-clockwise ordering with some arbitrary but henceforth fixed initial element a_1.

Let J_0 be the set of oriented simple-normal geodesic arcs γ on M such that the initial point and the ending point of γ lie in ∂R. (Note that except at its initial point or ending point γ cannot pass through a vertex of R.) For $\gamma \in J_0$, we call the components of $\gamma \cap R$ the *segments* of γ and the points of $\gamma \cap \partial R$ the partition points of γ. We label the partition points $t_0, t_1, \cdots, t_n$ in the order in which they occur along γ (note that we treat $t_i \in \partial R$ as the initial point of the segment of γ from t_i to t_{i+1}) and we set $\| \gamma \| = n$ as the combinatorial length of γ.

For $\gamma \in J_0$, the segments of γ give rise to a *simple diagram* on R which is a collection of finitely many pairwise disjoint (geodesic) arcs joining pairs of distinct elements of A. Two simple diagrams are regarded as being identical if they agree up to isotopy supported on each side of R. For $a_i, a_j \in A, i \neq j$, let n_{ij} denote the number of arcs joining a_i to a_j in the given simple diagram. The *length* of a simple diagram is $n = \sum n_{ij}, 1 \leq i < j \leq m$.

The Birman–Series parameterization of elements of J_0 consists of two sets of data. The first is the ordered sequence $h_1(\gamma) = (n_{12}, n_{13}, \cdots, n_{m-1,m})$ which records for each pair of distinct elements a_i, a_j of R the number n_{ij} of segments of γ which join a_i to a_j. The second set of data, $h_2(\gamma)$, records information about the position of the initial and final points t_0, t_n of γ. Let $a(t_i)$ be the element of A containing t_i and let $j(t_i) \in \mathbf{N}$ be the position of t_i among the partition points of γ which lie along $a(t_i)$ counting in the anticlockwise direction round ∂R. Define $h_2(\gamma) = (a(t_0), j(t_0), a(t_n), j(t_n))$.

The following lemmas and their proofs in [5] still hold in our case.

Lemma 7.3. *Suppose that $\gamma, \gamma' \in J_0$ and that $h_1(\gamma) = h_1(\gamma'), h_2(\gamma) = h_2(\gamma')$. Let $t_0, t_1, \cdots, t_n$ and $t'_0, t'_1, \cdots, t'_n$ be the partition points of γ, γ' respectively. Then $a(t_i) = a(t'_i)$ for each $i = 0, 1, \cdots, n$.*

Lemma 7.4. *Let $J_0(n) = \{\gamma \in J_0 : \| \gamma \| = n\}$. Then there is a polynomial $P_0(n)$ such that the number of simple diagrams of length n*

$$\mathrm{card}\{(h_1(\gamma), h_2(\gamma)) : \gamma \in J_0(n)\} \leq P_0(n).$$

The main idea of the proof of Birman–Series Theorem in [5] is that geodeisc arcs in $J_0(n)$ (for sufficiently large n) with the same parameterization lie exponentially close in M. It relies on the following key lemma which is Lemma 3.1 in [5].

Lemma 7.5. *There is a universal constant $\alpha > 0$ (depending only on the choice of the fundamental polygon R) so that*

$$l(\gamma) \geq \alpha \| \gamma \|$$

for $\gamma \in J_0$ with $\| \gamma \|$ sufficiently large, where $l(\gamma)$ denotes the hyperbolic length of γ.

Proof: There is a universal constant $\epsilon > 0$ so that any segment of γ which does not connect two consecutive sides of R or does not intersect a suitably chosen disk neighborhood of each cusp or cone point has hyperbolic length at least ϵ. Let q be the maximum number of sides of R, projected to M, which meet at any cusp or cone point of M. Then at most $q-1$ consecutive segments of γ can connect consecutive sides of R around the same cusp or cone point and intersect the chosen disk neighborhood of that cusp or cone point; for otherwise there will be a self-intersection on γ. Hence in any q consecutive segments of γ, at least one has hyperbolic length ϵ, which gives the result. $\square$

The following two lemmas then apply respectively to the set of all complete simple geodesics which never intersect any cusp or cone point and to the set of simple geodesics which start from a fixed cusp or cone point and never terminates at any cusp or cone point. (Recall that we assume that M has no boundary geodesics.)

Lemma 7.6. *Let $\gamma, \gamma' \in J_0(2n+1)$ and suppose that $h_1(\gamma) = h_1(\gamma'), h_2(\gamma) = h_2(\gamma')$. Let $\delta \subset \gamma, \delta' \subset \gamma'$ denote the segments of γ, γ' lying between the partition points t_n, t_{n+1} and t'_n, t'_{n+1} respectively. Then $\delta' \subset B_{ce^{-\alpha n}}(\delta)$ where c, α are universal constants and where $B_\epsilon(\delta)$ denotes the tubular neighborhood of δ of hyperbolic radius $\epsilon > 0$.*

Lemma 7.7. *Let $\gamma, \gamma' \in J_0(n+k)$ be such that they start at the same vertex of R and that $h_1(\gamma) = h_1(\gamma'), h_2(\gamma) = h_2(\gamma')$. Let $\delta \subset \gamma, \delta' \subset \gamma'$ denote the segments of γ, γ' lying between the partition points t_i, t_{i+1} and t'_i, t'_{i+1} respectively for some $1 \le i \le k$. Then $\delta' \subset B_{ce^{-\alpha n}}(\delta)$ where c, α are universal constants and where $B_\epsilon(\delta)$ denotes the tubular neighborhood of δ of hyperbolic radius $\epsilon > 0$.*

Note that Lemma 7.6 is Lemma 3.2 in [5] and Lemma 7.7 can be proved similarly.

From these we have the following proposition which is Proposition 4.1 in [5] from which the conclusion of the Birman–Series Theorem follows exactly as in the proofs in [5] §5.

Proposition 7.8. *There exist universal constants $L, c, \alpha > 0$ and a polynomial $P_0(\cdot)$ such that for each n there is a set F_n of simple geodesic arcs, each of length at most L, so that $\mathrm{card}(F_n) \le P_0(n)$ and so that*

$$S_0 \subset \cup\{B_\epsilon(\gamma) \mid \gamma \in F_n\}, \epsilon = ce^{\alpha n}.$$

Finally we remark that the above Birman–Series' arguments will give rough estimates on the distribution of simple closed geodesics on a compact hyperbolic cone-surface M which is enough for proving the absolute convergence of the series appearing in various generalized McShane's identities, as was observed and used in [1] (for the case of complete hyperbolic surfaces) for similar purposes.

Lemma 7.9. *Let M be a compact hyperbolic cone-surface with all cone angles in $(0, \pi]$. Then for any constant $c > 0$*

(i) *the series*

$$\sum_\beta \frac{1}{\exp(c|\beta|)}$$

converges absolutely, where the sum is over all generalized simple closed geodesics on M and all simple normal geodesic arcs connecting geometric boundary components of M;

(ii) *the series*

$$\sum_{\alpha,\beta} \frac{1}{\exp[c(|\alpha| + |\beta|)]}$$

converges absolutely, where the sum is over all pairs α, β of disjoint generalized simple closed geodesics on M and/or simple normal geodesic arcs connecting geometric boundary components of M. □

8. **Proof of Theorems**

Proof of Theorem 1.8 Now the proof is obvious from the previous discussions. Suppose Δ_0 is a cone point. Recall $\mathcal{H}$ is a suitably chosen small circle centered at Δ_0, and $\mathcal{H}_{\mathtt{ns}}$, $\mathcal{H}_{\mathtt{sn}}$, $\mathcal{H}_{\mathtt{snn}}$ are the point sets of the first intersections of $\mathcal{H}$ with respectively all non-simple, all simple-normal, all simple-not-normal Δ_0-geodesics. The elliptic measure of each of these subsets of $\mathcal{H}$ is the radian measure that it subtends to the cone point Δ_0. The generalized Birman–Series Theorem in §7 implies that the closed subset $\mathcal{H}_{\mathtt{sn}}$ has measure 0. Hence the open subset $\mathcal{H}_{\mathtt{ns}} \cup \mathcal{H}_{\mathtt{snn}}$ has full measure, that is, θ_0. Now the maximal open intervals of $\mathcal{H}_{\mathtt{ns}} \cup \mathcal{H}_{\mathtt{snn}}$, suitably combined, have measure $2\mathrm{Gap}(\Delta_0; \alpha, \beta)$ for each unordered pair of generalized simple closed geodesics α, β on M which bound with Δ_0 an embedded pair of pants on M. Hence their sum is equal to θ_0 and the desired identity follows. The cases where Δ_0 is a boundary geodesic or a cusp are similarly proved. $\square$

Proof of Corollary 1.10 Consider the case where Δ_0 is a cone point. In this case T admits a unique elliptic involution η such that η maps each oriented simple closed geodesics on T onto itself with orientation reversed. Note that η fixes the cone point Δ_0 and three other interior points which are the so-called Weierstrass points of T. Each simple closed geodesics on T passes exactly two Weierstrass points; hence there are three Weierstrass classes of simple closed geodesics on T. Now the quotient of T under η is a sphere with three angle π cone points and a cone point with angle $\theta/2$. Then Theorem 1.8 applies to $M = T/\langle\eta\rangle$, with Δ_0 the angle π cone point whose inverse image under η is the Weierstrass point that the Weierstrass class $\mathcal{A}$ misses. Note that each generalized simple closed geodesic on $M = T/\langle\eta\rangle$ is either a geometric boundary component or degenerate simple closed geodesic which is the double cover of a simple geodesic arc which connects two Weierstrass points. Hence the set of all pairs of generalized simple closed geodesics which bound with Δ_0 an embedded pair of pants is exactly the set of pairs consisting of the angle $\theta/2$ cone point plus a degenerate simple closed geodesic γ' which is the double cover of the quotient simple geodesic arc of a simple closed geodesic γ on T in the given Weierstrass class $\mathcal{A}$ (note that by definition the length of γ' is the same as that of γ). Hence by (26) the summand in the summation is

$$\frac{\pi}{2} - \tan^{-1}\left(\frac{\sin\frac{\pi}{2}\sinh\frac{|\gamma|}{2}}{\cos\frac{\theta}{4} + \cos\frac{\pi}{2}\cosh\frac{|\gamma|}{2}}\right) = \tan^{-1}\left(\frac{\cos\frac{\theta}{4}}{\sinh\frac{|\gamma|}{2}}\right).$$

The proof for the case where Δ_0 is a boundary geodesic is similar. $\square$

Remark 8.1. Note that we can also choose Δ_0 to be the angle $\theta/2$ cone point on $T/\langle\eta\rangle$, then we obtain (4), the generalization of McShane's original identity to the cone-torus T. This is one way of seeing why we can allow the cone angle of up to 2π in the cone torus case.

Proof of Theorem 1.13 It is well known that M admits a unique hyperelliptic involution η (see for example [14]) such that η maps each simple closed geodesic onto itself and preserves/reverses the orientation of separating/non-separating simple closed geodesics. Note that η leaves six points on M fixed; they are the six

Weierstrass points on M. Consider the quotient $M' = M/\langle\eta\rangle$ which is a sphere with six angle π cone points. Each generalized simple closed geodesic on M' is either

(i) an angle π cone point; or
(ii) a degenerate simple closed geodesic β' which is the double cover of a simple geodesic arc c connecting two angle π cone points where the inverse image of c under η is a non-separating simple closed geodesic β on M; or
(iii) a separating (non-degenerate) simple closed geodesic α' whose inverse image under η is a separating simple closed geodesic α on M. In this case α' does not pass through any of the six angle π cone points and there are three of them on each side of α' on M'. Hence α passes none of six Weierstrass points and there are three of them on each side of α on M.

Now apply Theorem 1.8 to M' with Δ_0 one of the six angle π cone points. Then each pair of generalized simple closed geodesics on M' which bound with Δ_0 an embedded pair of pants $\mathcal{P}$ consists of a separating simple closed geodesic α' on M' and a degenerate simple closed geodesic β' on M' which lies on the same side of α' as Δ_0 and misses Δ_0. Let the inverse image of α', β' under η be α, β. Then α is a separating simple closed geodesic on M and β is a non-separating simple closed geodesic on M. Furthermore, β and the Weierstrass point which is the inverse image of Δ_0 lie on the same side of α on M. Note that the hyperbolic lengths of α', β' are respectively $|\alpha|/2, |\beta|$. Hence by (24) in this case the summand in the resulting generalized McShane's Weierstrass identity for M' with the chosen Δ_0 is

$$2\tan^{-1}\left(\frac{\sin\frac{\pi}{2}}{\cos\frac{\pi}{2} + \exp\frac{|\alpha|/2+|\beta|}{2}}\right) = 2\tan^{-1}\exp\left(-\frac{|\alpha|}{4} - \frac{|\beta|}{2}\right).$$

Note that each pair of disjoint simple closed geodesics (α, β) on M such that α is separating and β is non-separating arises as the inverse image of a unique pair of generalized simple closed geodesics on M' as described above, where the chosen Δ_0 is the angle π cone point which is the image under η of the Weierstrass point on M that lies on the same side of α as β and is missed by β.

Summing all the six resulting Weierstrass identities we then have

$$\sum 2\tan^{-1}\exp\left(-\frac{|\alpha|}{4} - \frac{|\beta|}{2}\right) = \frac{6\pi}{2},$$

where the sum is over all ordered pairs (α, β) of disjoint simple closed geodesics on M such that α is separating and β is non-separating. $\square$

Proof of Addendum 1.15 We first prove that the series in (17) converges absolutely and uniformly on compact set in the space $\mathcal{QF}$ of quasi-Fuchsian representations of $\pi_1(M)$ into $\mathrm{SL}(2, \mathbf{C})$ by the same argument as used in [1]. The identity (17) then follows by analytic continuation since each summand in it is an analytic function of the complex Fenchel–Nielsen coordinates for the quasi-Fuchsian space (see [25]) and the identity holds when all the coordinates take real values (by Theorem 1.13) and the space of quasi-Fuchsian representations of $\pi_1(M)$ into $\mathrm{PSL}(2, \mathbf{C})$ is simply connected.

As pointed out in [1] Lemma 5.2, by [15] Lemma 3, for any compact subset $\mathcal{C}$ of $\mathcal{QF}$, there is a constant $k = k(C) > 0$ such that

$$k l_{\rho_0}(\gamma) \le \Re l_\rho(\gamma) \le k^{-1} l_{\rho_0}(\gamma),$$

for any essential simple closed curve γ, where ρ_0 is a fixed Fuchsian representation of $\pi_1(M)$ into $\mathrm{SL}(2,\mathbf{C})$.

Since $|\tan^{-1}(x)| \le 2|x|$ for $|x|$ sufficiently small, we have for all except a finitely many pairs of (free homotopy classes of) disjoint essential simple closed curves α, β on M such that α is separating and β is non-separating

$$\begin{aligned} \left|\tan^{-1}\exp\Big(-\frac{l_\rho(\alpha)}{4}-\frac{l_\rho(\beta)}{2}\Big)\right| &\le 2\left|\exp\Big(-\frac{l_\rho(\alpha)}{4}-\frac{l_\rho(\beta)}{2}\Big)\right| \\ &= 2\exp\Big(-\frac{\Re l_\rho(\alpha)}{4}-\frac{\Re l_\rho(\beta)}{2}\Big) \\ &\le 2\exp\Big(-k\Big(\frac{l_{\rho_0}(\alpha)}{4}+\frac{l_{\rho_0}(\beta)}{2}\Big)\Big). \end{aligned}$$

Thus the series in (17) converges absolutely and uniformly on the compact set C of $\mathcal{QF}$ since the series

$$\sum \exp\Big(-k\Big(\frac{l_{\rho_0}(\alpha)}{4}+\frac{l_{\rho_0}(\beta)}{2}\Big)\Big)$$

converges by Lemma 7.9. □

9. Complexified reformulation of the generalized McShane's identity

In this section we prove the unified version (18) of our generalized McShane's identity using complex arguments and interpret it geometrically.

Two functions First we would like to define two functions $G, S : \mathbf{C}^3 \to \mathbf{C}$ as follows:

$$G(x,y,z) = 2\tanh^{-1}\left(\frac{\sinh(x)}{\cosh(x)+\exp(y+z)}\right), \tag{58}$$

$$S(x,y,z) = \tanh^{-1}\left(\frac{\sinh(x)\sinh(y)}{\cosh(z)+\cosh(x)\cosh(y)}\right). \tag{59}$$

Note that here for a complex number x, $\tanh^{-1}(x)$ is defined to have imaginary part in $(-\pi/2, \pi/2]$. Using the identity

$$x = \frac{1}{2}\log\frac{1+\tanh(x)}{1-\tanh(x)},$$

it is easy to check that the two functions have also the following expressions:

$$G(x,y,z) = \log\frac{\exp(x)+\exp(y+z)}{\exp(-x)+\exp(y+z)}, \tag{60}$$

$$S(x,y,z) = \frac{1}{2}\log\frac{\cosh(z)+\cosh(x+y)}{\cosh(z)+\cosh(x-y)}, \tag{61}$$

as used by Mirzakhani in [21]. (She uses different notations $\mathcal{D}, \mathcal{R}$ as explained below.) Here for a non-zero complex number x, $\log(x)$ assumes the main branch

value with imaginary part in $(-\pi, \pi]$. We shall see that both expressions of the functions are useful.

For $x, y, z > 0$, the geometrical meanings of $G(x, y, z)$ and $S(x, y, z)$ are as follows. Let $\mathcal{P}(2x, 2y, 2z)$ be the unique hyperbolic pair of pants whose boundary components X, Y, Z are simple closed geodesics of lengths $2x, 2y, 2z$ respectively. Then $S(x, y, z)$ is half the length of the orthogonal projection of the boundary geodesic Y onto X in $\mathcal{P}(2x, 2y, 2z)$ and $S(x, z, y)$ is half the length of the orthogonal projection of the boundary geodesic Z onto X in $\mathcal{P}(2x, 2y, 2z)$, and $G(x, y, z)$ is the length of each of the two gaps between these two projections on X. We have therefore the identity

$$G(x, y, z) + S(x, y, z) + S(x, z, y) = x \tag{62}$$

for all $x, y, z \geq 0$. Note that the same identity holds modulo πi for all $x, y, z \in \mathbf{C}$.

Remark 9.1. The relations of our functions G, S with Mirzakhani's functions $\mathcal{D}, \mathcal{R}$ are

$$G(x, y, z) = \mathcal{D}(2x, 2y, 2z)/2, \tag{63}$$

$$S(x, y, z) = (x - \mathcal{R}(2x, 2z, 2y))/2. \tag{64}$$

Lemma 9.2. (i) *For $x, z \geq 0$ and $y \in [0, \frac{\pi}{2}]$,*

$$G(x, yi, z) + S(x, yi, z) = x - \tanh^{-1}\left(\frac{\sinh(x)\sinh(z)}{\cos(y) + \cosh(x)\cosh(z)}\right). \tag{65}$$

(ii) *For $x, y \in [0, \frac{\pi}{2}]$ and $z \geq 0$,*

$$G(xi, yi, z) + S(xi, yi, z) = \left[x - \tan^{-1}\left(\frac{\sin(x)\sinh(z)}{\cos(y) + \cos(x)\cosh(z)}\right)\right] i. \tag{66}$$

Proof: (i) It follows from the following two identities since $\Re S(x, yi, z) = 0$:

$$\Re G(x, yi, z) = x - \tanh^{-1}\left(\frac{\sinh(x)\sinh(z)}{\cos(y) + \cosh(x)\cosh(z)}\right), \tag{67}$$

$$\Im G(x, yi, z) + \Im S(x, yi, z) = 0. \tag{68}$$

Proof of (67) *and* (68): By definition,

$$\begin{aligned} G(x, yi, z) &= \log \frac{\exp(x) + \exp(yi + z)}{\exp(-x) + \exp(yi + z)} \\ &= \log \frac{[\exp(x) + \cos(y)\exp(z)] + i[\sin(y)\exp(z)]}{[\exp(-x) + \cos(y)\exp(z)] + i[\sin(y)\exp(z)]}. \end{aligned}$$

Hence

$$
\begin{aligned}
\Re G(x,yi,z) &= \frac{1}{2}\log\frac{[\exp(x)+\cos(y)\exp(z)]^2+[\sin(y)\exp(z)]^2}{[\exp(-x)+\cos(y)\exp(z)]^2+[\sin(y)\exp(z)]^2}\\
&= \frac{1}{2}\log\frac{\exp(2x)+\exp(2z)+2\exp(x)\cos(y)\exp(z)}{\exp(-2x)+\exp(2z)+2\exp(-x)\cos(y)\exp(z)}\\
&= \frac{1}{2}\log\left(\frac{\cosh(x-z)+\cos(y)}{\cosh(x+z)+\cos(y)}\frac{\exp(x+z)}{\exp(-x+z)}\right)\\
&= x-\frac{1}{2}\log\frac{\cosh(x+z)+\cos(y)}{\cosh(x-z)+\cos(y)}\\
&= x-\tanh^{-1}\left(\frac{\sinh(x)\sinh(z)}{\cos(y)+\cosh(x)\cosh(z)}\right).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\Im G(x,yi,z) &= \tan^{-1}\left(\frac{\sin(y)\exp(z)}{\exp(x)+\cos(y)\exp(z)}\right)-\tan^{-1}\left(\frac{\sin(y)\exp(z)}{\exp(-x)+\cos(y)\exp(z)}\right)\\
&= \tan^{-1}\left(\frac{[\exp(-x)-\exp(x)]\sin(y)\exp(z)}{[\exp(x)+\cos(y)\exp(z)][\exp(-x)+\cos(y)\exp(z)]+[\sin(y)\exp(z)]^2}\right)\\
&= \tan^{-1}\left(\frac{[\exp(-x)-\exp(x)]\sin(y)\exp(z)}{1+\exp(2z)+[\exp(x)+\exp(-x)]\cos(y)\exp(z)}\right)\\
&= -\tan^{-1}\left(\frac{\sinh(x)\sin(y)}{\cosh(z)+\cosh(x)\cos(y)}\right)\\
&= -\Im S(x,yi,z),
\end{aligned}
$$

since

$$
\begin{aligned}
S(x,yi,z) &= \tanh^{-1}\left(\frac{\sinh(x)\sinh(yi)}{\cosh(z)+\cosh(x)\cosh(yi)}\right)\\
&= i\,\tan^{-1}\left(\frac{\sinh(x)\sin(y)}{\cosh(z)+\cosh(x)\cos(y)}\right).
\end{aligned}
$$

(ii) It will follow from the following two identities:

$$\Im G(xi,yi,z)=x-\tan^{-1}\left(\frac{\sin(x)\sinh(z)}{\cos(y)+\cos(x)\cosh(z)}\right), \tag{69}$$

$$\Re G(xi,yi,z)+S(xi,yi,z)=0. \tag{70}$$

Proof of (69) *and* (70): By definition,

$$
\begin{aligned}
G(xi,yi,z) &= \log\frac{\exp(xi)+\exp(yi+z)}{\exp(-xi)+\exp(yi+z)}\\
&= \log\frac{[\cos(x)+\cos(y)\exp(z)]+i[\sin(x)+\sin(y)\exp(z)]}{[\cos(x)+\cos(y)\exp(z)]+i[-\sin(x)+\sin(y)\exp(z)]}.
\end{aligned}
$$

Hence

$$\begin{aligned}
\Re G(xi, yi, z) &= \frac{1}{2}\log\frac{[\cos(x)+\cos(y)\exp(z)]^2+[\sin(x)+\sin(y)\exp(z)]^2}{[\cos(x)+\cos(y)\exp(z)]^2+[-\sin(x)+\sin(y)\exp(z)]^2}\\
&= \frac{1}{2}\log\frac{1+\exp(2z)+\cos(x-y)\exp(z)}{1+\exp(2z)+\cos(x+y)\exp(z)}\\
&= \frac{1}{2}\log\frac{\cosh(z)+\cos(x-y)}{\cosh(z)+\cos(x+y)}\\
&= -\frac{1}{2}\log\frac{\cosh(z)+\cosh(xi+yi)}{\cosh(z)+\cosh(xi-yi)}\\
&= -S(xi, yi, z).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
I &= \Im G(xi, yi, z)\\
&= \tan^{-1}\left(\frac{\sin(x)+\sin(y)\exp(z)}{\cos(x)+\cos(y)\exp(z)}\right)-\tan^{-1}\left(\frac{-\sin(x)+\sin(y)\exp(z)}{\cos(x)+\cos(y)\exp(z)}\right)\\
&= \tan^{-1}\left(\frac{2\sin(x)[\cos(x)+\cos(y)\exp(z)]}{[\cos(x)+\cos(y)\exp(z)]^2-[\sin(x)]^2+[\sin(y)\exp(z)]^2}\right)\\
&= \tan^{-1}\left(\frac{\sin(2x)+2\sin(x)\cos(y)\exp(z)}{\cos(2x)+\exp(2z)+2\cos(x)\cos(y)\exp(z)}\right).
\end{aligned}$$

Hence

$$iI = \tanh^{-1}\left(\frac{i\sin(2x)+2i\sin(x)\cos(y)\exp(z)}{\cos(2x)+\exp(2z)+2\cos(x)\cos(y)\exp(z)}\right),$$

or

$$\frac{\exp(2iI)-1}{\exp(2iI)+1} = \frac{i\sin(2x)+2i\sin(x)\cos(y)\exp(z)}{\cos(2x)+\exp(2z)+2\cos(x)\cos(y)\exp(z)}.$$

Hence

$$\begin{aligned}
\exp(2iI) &= \frac{\exp(2xi)+\exp(2z)+2\exp(xi)\cos(y)\exp(z)}{\exp(-2xi)+\exp(2z)+2\exp(-xi)\cos(y)\exp(z)}\\
&= \frac{\cosh(xi-z)+\cos(y)}{\cosh(xi+z)+\cos(y)}\frac{\exp(xi+z)}{\exp(-xi+z)}\\
&= \exp(2xi)\frac{\cosh(yi)+\cosh(xi-z)}{\cosh(yi)+\cosh(xi+z)}.
\end{aligned}$$

Thus

$$\begin{aligned}
iI &= xi-\frac{1}{2}\log\frac{\cosh(yi)+\cosh(xi+z)}{\cosh(yi)+\cosh(xi-z)}\\
&= xi-\tanh^{-1}\left(\frac{\sinh(xi)\sinh(z)}{\cosh(yi)+\cosh(xi)\cosh(z)}\right)\\
&= xi-i\tan^{-1}\left(\frac{\sin(x)\sinh(z)}{\cos(y)+\cos(x)\cosh(z)}\right).
\end{aligned}$$

Therefore

$$\Im G(xi, yi, z) = I = x-\tan^{-1}\left(\frac{\sin(x)\sinh(z)}{\cos(y)+\cos(x)\cosh(z)}\right).$$

$\square$

Restatement of the complexified identities Now we can restate the non-cusp cases of Theorem 1.16 using the functions G, S defined above. Recall that for each generalized simple closed geodesic δ, we have defined in §1 its **complex length** $|\delta|$, that is, $|\delta| = 0$ if δ is a cusp; $|\delta| = \theta i$ if δ is a cone point of angle $\theta \in (0, \pi]$; and $|\delta| = l$ if δ is a boundary geodesic or an interior generalized simple closed geodesic of hyperbolic length $l > 0$.

Theorem 9.3. *For a compact hyperbolic cone-surface M with all cone angles in $(0, \pi]$, let all its geometric boundary components be $\Delta_0, \Delta_1, \cdots, \Delta_n$ with complex lengths $L_0, L_1, \cdots, L_n$ respectively. If Δ_0 is a cone point or a boundary geodesic then*

$$\sum_{\alpha,\beta} G\left(\frac{L_0}{2}, \frac{|\alpha|}{2}, \frac{|\beta|}{2}\right) + \sum_{j=1}^{n} \sum_{\beta} S\left(\frac{L_0}{2}, \frac{L_j}{2}, \frac{|\beta|}{2}\right) = \frac{L_0}{2}, \tag{71}$$

where the first sum is over all (unordered) pairs of generalized simple closed geodesics α, β on M such that α, β bound with Δ_0 an embedded pair of pants on M (note that one of α, β might be a geometric boundary component) and the sub-sum in the second sum is over all interior simple closed geodesics β such that β bounds with Δ_j and Δ_0 an embedded pair of pants on M. Furthermore, all the series in (71) converge absolutely.

Remark 9.4. We shall omit the proof of Theorem 1.16 in the case where Δ_0 is a cusp, for as remarked before, in the cusp case the identity (19) can either be proved similarly or be derived by considering the first order infinitesimal terms of the corresponding identity (18) in other cases.

Proof: We first show that our generalized McShane's identities (8) and (9) can be reformulated as (71) modulo convergence.

First suppose that Δ_0 is a boundary geodesic of hyperbolic length $l_0 > 0$.

For a pair of interior generalized simple closed geodesics α, β which bound with Δ_0 an embedded pair of pants on M, we have directly by definition that

$$\mathrm{Gap}(\Delta_0; \alpha, \beta) = G\left(\frac{l_0}{2}, \frac{|\alpha|}{2}, \frac{|\beta|}{2}\right).$$

For a pair of generalized simple closed geodesics α, β such that α is a boundary geodesic and β is an interior generalized simple closed geodesic and that they bound with Δ_0 an embedded pair of pants on M, we have by definition and the geometric meanings of G, S that

$$\mathrm{Gap}(\Delta_0; \alpha, \beta) = G\left(\frac{l_0}{2}, \frac{|\alpha|}{2}, \frac{|\beta|}{2}\right) + S\left(\frac{l_0}{2}, \frac{|\alpha|}{2}, \frac{|\beta|}{2}\right).$$

For a pair of generalized simple closed geodesics α, β such that α is a cone point of angle $\varphi \in (0, \pi]$ and β is an interior generalized simple closed geodesic and that they bound with Δ_0 an embedded pair of pants on M, we have by (65) with $x = l_0/2, y = \varphi/2, z = |\beta|/2$ that

$$\mathrm{Gap}(\Delta_0; \alpha, \beta) = G\left(\frac{l_0}{2}, \frac{\varphi i}{2}, \frac{|\beta|}{2}\right) + S\left(\frac{l_0}{2}, \frac{\varphi i}{2}, \frac{|\beta|}{2}\right).$$

Next suppose that Δ_0 is a cone point of angle $\theta_0 \in (0, \pi]$.

For a pair of interior generalized simple closed geodesics α, β which bound with Δ_0 an embedded pair of pants on M, we have by definition that

$$\mathrm{Gap}(\Delta_0; \alpha, \beta)\, i = G\left(\frac{\theta_0 i}{2}, \frac{|\alpha|}{2}, \frac{|\beta|}{2}\right).$$

For a pair of generalized simple closed geodesics α, β such that α is a boundary geodesic and β is an interior generalized simple closed geodesic and that they bound with Δ_0 an embedded pair of pants on M, we have by the analysis in §6 that

$$\begin{aligned} & \mathrm{Gap}(\Delta_0; \alpha, \beta)\, i \\ = \;& 2i \tan^{-1} \frac{\sin \frac{\theta_0}{2}}{\cos \frac{\theta_0}{2} + \exp \frac{|\alpha|+|\beta|}{2}} + i \tan^{-1} \frac{\sin \frac{\theta_0}{2} \sinh \frac{|\alpha|}{2}}{\cosh \frac{|\beta|}{2} + \cos \frac{\theta_0}{2} \cosh \frac{|\alpha|}{2}} \\ = \;& G\left(\frac{\theta_0 i}{2}, \frac{|\alpha|}{2}, \frac{|\beta|}{2}\right) + S\left(\frac{\theta_0 i}{2}, \frac{|\alpha|}{2}, \frac{|\beta|}{2}\right). \end{aligned}$$

For a pair of generalized simple closed geodesics α, β such that α is a cone point of angle $\varphi \in (0, \pi]$ and β is an interior generalized simple closed geodesic and that they bound with Δ_0 an embedded pair of pants on M, we have by (66) with $x = \theta_0/2, y = \varphi/2, z = |\beta|/2$ that

$$\mathrm{Gap}(\Delta_0; \alpha, \beta)\, i = G\left(\frac{\theta_0 i}{2}, \frac{\varphi i}{2}, \frac{|\beta|}{2}\right) + S\left(\frac{\theta_0 i}{2}, \frac{\varphi i}{2}, \frac{|\beta|}{2}\right).$$

Finally we prove the absolute convergence of the series in (71). It is not hard to see that we only need to prove, for each $j = 1, \cdots, n$, the absolute convergence of the series

$$\sum_{\beta} S\left(\frac{L_0}{2}, \frac{L_j}{2}, \frac{|\beta|}{2}\right),$$

where the sum is over all interior generalized simple closed geodesics β which bounds with Δ_j and Δ_0 an embedded pair of pants on M. The desired absolute convergence follows from Lemma 7.9 since

$$S\left(\frac{L_0}{2}, \frac{L_j}{2}, \frac{|\beta|}{2}\right) \sim \frac{\sinh \frac{L_0}{2} \sinh \frac{L_j}{2}}{\cosh \frac{|\beta|}{2}} \sim \text{const.} \exp\left(-\frac{|\beta|}{2}\right)$$

as $|\beta| \to \infty$. □

Geometric interpretation We would like to explore the geometric meanings of the summands in the complexified formula (71).

In the case that M has no cone points, all its geometric boundary components (here cusps are treated as boundary geodesics of length 0) $\Delta_0, \Delta_1, \cdots, \Delta_n$ are boundary geodesics with hyperbolic lengths $L_0, L_1, \cdots, L_n$ respectively. Assume Δ_0 is not a cusp, that is, $L_0 > 0$. Then as explained in §6, in the first sum the summand is the width of one of the main gaps in the pair of pants $\mathcal{P}(\Delta_0, \alpha, \beta)$ bounded by Δ_0 and α, β; while in the second sum the sub-summand is the width of one of the two extra gaps associated to Δ_j in the pair of pants $\mathcal{P}(\Delta_0, \Delta_j, \beta)$

bounded by Δ_0, Δ_j and β. We would like to think of the union of the two extra gaps in $\mathcal{P}(\Delta_0, \Delta_j, \beta)$ as the orthogonal projection of Δ_i onto Δ_0 along the common perpendicular δ of Δ_j and Δ_0 in $\mathcal{P}(\Delta_0, \Delta_j, \beta)$ and think of its width as the *direct visual measure* of Δ_j at Δ_0 along δ. Hence the second part of the left hand side of (71) can be thought of as the total direct visual measure of all the non-distinguished geometric boundary components $\Delta_1, \cdots, \Delta_n$ at Δ_0.

In the case that Δ_0 is a cone point of angle $\theta_0 \in (0, \pi]$ (hence $L_0 = \theta_0 i$) and all other geometric boundary components of M are boundary geodesics (here cusps treated as boundary geodesics of length 0), for each pair of generalized simple closed geodesics α, β which bound with Δ_0 an embedded pair of pants $\mathcal{P}(\Delta_0, \alpha, \beta)$ on M, each of α, β has a direct visual angle at the cone point Δ_0; and the summand in the first sum is i times the angle measure of one of the two gaps at Δ_0 between the two Δ_0-geodesic rays asymptotic to α^+, β^-(respectively α^-, β^+). The sub-summand in the second sum is i times half the visual angle measure of Δ_j at Δ_0 in the pair of pants $\mathcal{P}(\Delta_0, \Delta_j, \beta)$ on M.

When M has cone points other than Δ_0, the similar formulations of the generalized McShane's identities (8)–(10) in terms of $\mathrm{Gap}(\Delta_0; \alpha, \beta)$ will not be as neat as in the above two special cases. The problem lies in that a cone point (other than Δ_0) *seems* to have direct visual measure zero at Δ_0, causing the formulas to be non-uniform. However, this non-uniformity is caused by the (wrong) point of view that we treat a cone point as only a point. The correct point of view is (perhaps) that a cone point (as a geometric boundary component) should be a geodesic perpendicular to the surface at the very cone point when the surface is "imagined" as lying in the hyperbolic 3-space and hence one should use purely complex length instead of real one for a cone point. (The point of view of using complex translation length for an isometry of the hyperbolic 3-space is well discussed in details in [11] and [24].)

First assume that Δ_0 is boundary geodesic of length $l_0 > 0$ and consider a pair of generalized simple closed geodesics α, β on M such that α is a cone point of angle $\varphi \in (0, \pi]$ and β is an interior generalized simple closed geodesic and that they bound with Δ_0 an embedded pair of pants $\mathcal{P}(\Delta_0, \alpha, \beta)$ on M.

Let the (unoriented) geodesic arc in $\mathcal{P}(\Delta_0, \alpha, \beta)$ which is perpendicular to Δ_0 and α (respectively, α and β, β and Δ_0) be denoted $[\Delta_0, \alpha]$ (respectively, $[\alpha, \beta]$, $[\beta, \Delta_0]$). We cut $\mathcal{P}(\Delta_0, \alpha, \beta)$ open along $[\Delta_0, \alpha]$, $[\alpha, \beta]$, $[\beta, \Delta_0]$ to obtain two congruent pentagons; lift one of them to a pentagon $\mathbf{P}(\Delta_0, \alpha, \beta)$ in the hyperbolic plane H^2. Then by Fenchel [11] $\mathbf{P}(\Delta_0, \alpha, \beta)$ can be regarded as a right angled hexagon $\mathbf{H}(\Delta_0, \tilde{\alpha}, \beta)$ spanned by straight lines $\Delta_0, \tilde{\alpha}, \beta$ in a hyperbolic 3-space H^3 containing the hyperbolic plane H^2. See Figure 9 for an illustration. Here $\tilde{\alpha}$ is the straight line in H^3 which passes through the cone point α in H^2 and is perpendicular to H^2. Let the common perpendiculars in H^3 between pairs of $\Delta_0, \tilde{\alpha}, \beta$ be $[\Delta_0, \tilde{\alpha}]$, $[\tilde{\alpha}, \beta]$, $[\beta, \Delta_0]$, where, as straight lines $[\Delta_0, \tilde{\alpha}]$, $[\tilde{\alpha}, \beta]$ are the same as $[\Delta_0, \alpha]$, $[\alpha, \beta]$ respectively.

We orient the six straight lines in the cyclic order $\Delta_0, [\Delta_0, \tilde{\alpha}], \tilde{\alpha}, [\tilde{\alpha}, \beta], \beta, [\beta, \Delta_0]$ as Fenchel did in [11]; see Figure 9. Then the three oriented sides $\Delta_0, \tilde{\alpha}, \beta$ of the right angled hexagon $\mathbf{H}(\Delta_0, \tilde{\alpha}, \beta)$ have complex lengths $\frac{l_0}{2} + \pi i, \frac{\varphi i}{2} + \pi i, \frac{|\beta|}{2} + \pi i$ respectively.

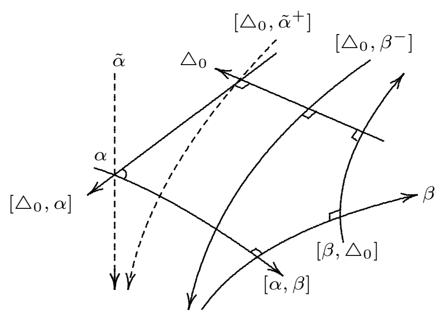

FIGURE 8

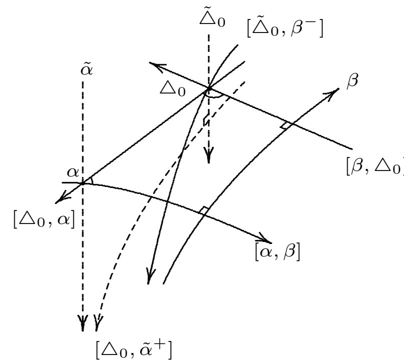

FIGURE 9

Let the ideal points which are the starting and ending endpoints of an oriented straight line $\mathbf{l}$ in H^3 be denoted $\mathbf{l}^-, \mathbf{l}^+$ respectively. Then we have in H^3 an oriented straight line $[\Delta_0, \tilde{\alpha}^+]$ which intersects Δ_0 perpendicularly and has $\tilde{\alpha}^+$ as its ending ideal point, and similarly an oriented straight line $[\Delta_0, \beta^-]$ which intersects Δ_0 perpendicularly and has β^- as its ending ideal point.

Then it can be verified that

$$G\left(\frac{l_0}{2}, \frac{\varphi i}{2} + \pi i, \frac{|\beta|}{2} + \pi i\right) = G\left(\frac{l_0}{2}, \frac{\varphi i}{2}, \frac{|\beta|}{2}\right)$$

is the complex length from $[\Delta_0, \beta^-]$ to $[\Delta_0, \tilde{\alpha}^+]$ measured along Δ_0 and

$$S\left(\frac{l_0}{2}, \frac{\varphi i}{2} + \pi i, \frac{|\beta|}{2} + \pi i\right) = S\left(\frac{l_0}{2}, \frac{\varphi i}{2}, \frac{|\beta|}{2}\right)$$

is the the complex length from $[\Delta_0, \tilde{\alpha}^+]$ to $[\Delta_0, \tilde{\alpha}]$ measured along Δ_0.

Note that $S(\frac{l_0}{2}, \frac{\varphi i}{2}, \frac{|\beta|}{2})$ is purely imaginary, which is obvious from its geometric meaning.

Remark 9.5. We remark that it is crucial that in $G(\frac{l_0}{2}, \frac{\varphi i}{2} + \pi i, \frac{|\beta|}{2} + \pi i)$ and $S(\frac{l_0}{2}, \frac{\varphi i}{2} + \pi i, \frac{|\beta|}{2} + \pi i)$ the value used for Δ_0 is $\frac{l_0}{2}$ instead of $\frac{l_0}{2} + \pi i$.

Next assume Δ_0 is a cone point of angle $\theta_0 \in (0, \pi]$ and consider a pair of generalized simple closed geodesics α, β on M such that α is a cone point of angle $\varphi \in (0, \pi]$ and β is an interior generalized simple closed geodesic and that they bound with Δ_0 an embedded pair of pants $\mathcal{P}(\Delta_0, \alpha, \beta)$ on M.

In this case we cut $\mathcal{P}(\Delta_0, \alpha, \beta)$ open along $[\Delta_0, \alpha]$, $[\alpha, \beta]$, $[\beta, \Delta_0]$ to obtain two congruent quadrilaterals and lift one of them to a quadrilateral $\mathbf{Q}(\Delta_0, \alpha, \beta)$ in the hyperbolic plane H^2. As before, let $\tilde{\alpha}$ be the straight line in H^3 which passes the cone point α in H^2 and is perpendicular to H^2. Similarly for $\tilde{\Delta}_0$. Then we obtain a right angled hexagon $\mathbf{H}(\tilde{\Delta}_0, \tilde{\alpha}, \beta)$ in H^3. Let the six sides of $\mathbf{H}(\tilde{\Delta}_0, \tilde{\alpha}, \beta)$ be oriented as illustrated in Figure 9. Then the three oriented sides $\tilde{\Delta}_0, \tilde{\alpha}, \beta$ of the right angled hexagon $\mathbf{H}(\tilde{\Delta}_0, \tilde{\alpha}, \beta)$ have complex lengths $\frac{\theta_0 i}{2} + \pi i, \frac{\varphi}{2} i + \pi i, \frac{|\beta|}{2} + \pi i$ respectively.

Similarly, we have in H^3 an oriented straight line $[\tilde{\Delta}_0, \tilde{\alpha}^+]$ which intersects $\tilde{\Delta}_0$ perpendicularly and has $\tilde{\alpha}^+$ as its ending ideal point, and another oriented straight

line $[\tilde{\Delta_0}, \beta^-]$ which intersects $\tilde{\Delta_0}$ perpendicularly and has β^- as its ending ideal point.

Then
$$G\left(\frac{\theta_0 i}{2}, \frac{\varphi i}{2}+\pi i, \frac{|\beta|}{2}+\pi i\right) = G\left(\frac{\theta_0 i}{2}, \frac{\varphi i}{2}, \frac{|\beta|}{2}\right)$$
is the complex length from $[\tilde{\Delta_0}, \beta^-]$ to $[\tilde{\Delta_0}, \tilde{\alpha}^+]$ measured along $\tilde{\Delta_0}$ and
$$S\left(\frac{\theta_0 i}{2}, \frac{\varphi i}{2}+\pi i, \frac{|\beta|}{2}+\pi i\right) = S\left(\frac{\theta_0 i}{2}, \frac{\varphi i}{2}, \frac{|\beta|}{2}\right)$$
is the the complex length from $[\tilde{\Delta_0}, \tilde{\alpha}^+]$ to $[\tilde{\Delta_0}, \tilde{\alpha}]$ measured along $\tilde{\Delta_0}$.

Note that $S(\frac{\theta_0 i}{2}, \frac{\varphi i}{2}+\pi i, \frac{|\beta|}{2}+\pi i)$ is real, which is obvious from its geometric meaning.

Remark 9.6. Here it is crucial that in $G(\frac{\theta_0 i}{2}, \frac{\varphi i}{2}+\pi i, \frac{|\beta|}{2}+\pi i)$ and $S(\frac{\theta_0 i}{2}, \frac{\varphi i}{2}+\pi i, \frac{|\beta|}{2}+\pi i)$ the value used for Δ_0 is $\frac{\theta_0 i}{2}$ instead of $\frac{\theta_0 i}{2}+\pi i$.

References

[1] Hirotaka Akiyoshi, Hideki Miyachi and Makoto Sakuma, *Variations of McShane's identity for punctured surface groups*, preprint.

[2] Hirotaka Akiyoshi, Hideki Miyachi and Makoto Sakuma, *A refinement of McShane's identity for quasifuchsian punctured torus groups*, preprint.

[3] Ara Basmajian, *The orthogonal spectrum of a hyperbolic manifold*, Amer. J. Math. **115** (1993), no. 5, 1139–1159.

[4] Alan F. Beardon, *The Geometry of Discrete Groups*, Graduate Texts in Mathematics **91**, Springer-Verlag, New York, 1983.

[5] Joan S. Birman and Caroline M. Series, *Geodesics with bounded intersection number on surfaces are sparsely distributed*, Topology **24** (1985), no. 2, 217–225.

[6] Brian H. Bowditch, *A proof of McShane's identity via Markoff triples*, Bull. London Math. Soc. **28** (1996), no. 1, 73–78.

[7] Brian H. Bowditch, *A variation of McShane's identity for once-punctured torus bundles*, Topology **36** (1997), no. 2, 325–334.

[8] Brian H. Bowditch, *Markoff triples and quasi-Fuchsian groups*, Proc. London Math. Soc. (3) **77** (1998), no. 3, 697–736.

[9] Peter Buser, *Geometry and Spectra of Compact Riemann Surfaces*, Birkhäuser, Boston, 1992.

[10] Daryl Cooper, Craig D. Hodgson and Steve P. Kerckhoff, *Three-dimensional orbifolds and cone-manifolds*, MSJ Memoirs, 5, Mathematical Society of Japan, Tokyo, 2000.

[11] Werner Fenchel, *Elementary Geometry in Hyperbolic Space*, Walter de Gruyter & Co., Berlin, 1989.

[12] William M. Goldman, *The modular group action on real* SL(2)*-characters of a one-holed torus*, Geom. Topol. **7** (2003), 443–486.

[13] Andrew Haas, *Diophantine approximation on hyperbolic Riemann surfaces*, Acta Math. **156** (1986), no. 1-2, 33–82.

[14] Andrew Haas and Perry Susskind, *The geometry of the hyperelliptic involution in genus two*, Proc. Amer. Math. Soc. **105** (1989), no. 1, 159–165.

[15] Troels Jørgensen and Albert Marden, *Two doubly degenerate groups*, Quart. J. Math. Oxford (2) **30** (1979), no. 118, 143–156.

[16] Greg McShane, *A remarkable identity for lengths of curves*, Ph.D. Thesis, University of Warwick, 1991.

[17] Greg McShane, *Simple geodesics and a series constant over Teichmuller space*, Invent. Math. **132** (1998), no. 3, 607–632.

[18] Greg McShane, *Weierstrass points and simple geogesics*, Bull. London Math. Soc. **36** (2004), no. 2, 181–187.
[19] Greg McShane, *Length series on Teichmuller space*, arXiv:math.GT/0403041 v1.
[20] Greg McShane, *Simple geodesics on surfaces of genus 2*, preprint.
[21] Maryam Mirzakhani, *Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces*, preprint.
[22] Makoto Sakuma, *Variations of McShane's identity for the Riley slice and 2-bridge links*, Hyperbolic spaces and related topics (Japanese) (Kyoto, 1998), Surikaisekikenkyusho Kokyuroku No. 1104 (1999), 103–108.
[23] Paul Schmutz Schaller, *Geometry of Riemann surfaces based on closed geodesics*, Bull. Amer. Math. Soc. (N.S.) **35** (1998), no. 3, 193–214.
[24] Caroline Series, *An extension of Wolpert's derivative formula*, Pacific J. Math. **197** (2001), no. 1, 223–239.
[25] Ser Peow Tan, *Complex Fenchel–Nielsen coordinates for quasi-Fuchsian structures*, Internat. J. Math. **5** (1994), no. 2, 239–251.
[26] Ser Peow Tan, Yan Loi Wong and Ying Zhang, *Schottky groups, generalized Markoff maps and McShane's identity*, in preparation.
[27] Ying Zhang, *Hyperbolic cone-surfaces, generalized Markoff maps, Schottky groups and McShane's identity*, Ph.D. Thesis, National University of Singapore, July 2004.

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